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ON SOME TOPOLOGICAL PROPERTIES FOR SPECIAL CLASSES OF BANACH SPACES. PART 2

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We consider some classes of infinite-dimensional Banach spaces with integrable derivatives. A compactness lemma for nonreflexive spaces is obtained. However some main topological properties for the given spaces are obtained.

This work is continuation of [1].

Theorem 1. $W_0^* \subset C(S;H)$ with continuous embedding. Moreover, for every $y, \xi \in W_0^*$ and $s, t \in S$ the next formula of integration by parts takes place

$$(y(t),\xi(t)) - (y(s),\xi(s)) = \int_s^t \{(y'(\tau),\xi(\tau)) + (y(\tau),\xi'(\tau))\}d\tau. \qquad (1)$$

In particular, when $y = \xi$ we have:

$$\frac{1}{2}(\|y(t)\|_H^2 - \|y(s)\|_H^2) = \int_s^t (y'(\tau), y(\tau))d\tau .$$

Proof. To simplify the proof we consider $S = [a,b]$ for some

$$-\infty < a < b < +\infty.$$

The validity of formula (1) for $y, \xi \in C^1(S;V)$ is checked by direct calculation. Now let $\varphi \in C^1(S)$ be such fixed that $\varphi(a) = 0$ and $\varphi(b) = 1$. Moreover, for $y \in C^1(S;V)$ let $\xi = \varphi y$ and $\eta = y - \varphi y$. Then, due to (1):

$$(\xi(t), y(t)) = \int_a^t \{\varphi'(s)(y(s), y(s)) + 2\varphi(s)(y'(s), y(s))\}ds ,$$

$$-(\eta(t), y(t)) = \int_t^b \{-\varphi'(s)(y(s), y(s)) + 2(1-\varphi(s))(y'(s), y(s))\}ds ,$$

from here for $\xi_i \in L_{q_i}(S;V_i^*)$ and $\eta_i \in L_{r_i'}(S;H)$ $(i = 1,2)$ such that $y' = = \xi_1 + \xi_2 + \eta_1 + \eta_2$ it follows:

$$\|y(t)\|_H^2 = \int_t^b \{\varphi'(s)(y(s), y(s)) + 2\varphi(s)(y'(s), y(s))\}ds - 2\int_t^b (y'(s), y(s))ds \le$$

$$\le \max_{s\in S} |\varphi'(s)| \cdot \|y\|_{C(S;V^*)} \cdot \|y\|_{L_1(S;V)} + 2\int_S (\varphi(s)-1)(y'(s), y(s))ds \le$$

$$\le \max_{s\in S} |\varphi'(s)| \|y\|_{C(S;V^*)} \|y\|_{L_1(S;V)} +$$

$$+ 2\max_{s\in S} |\varphi(s)-1| \Big(\|\xi_1\|_{L_{q_1}(S;V_1^*)} \|y\|_{L_{p_1}(S;V_1)} + \|\xi_2\|_{L_{q_2}(S;V_2^*)} \|y\|_{L_{p_2}(S;V_2)} +$$

$$+ \|\eta_1\|_{L_{r_1'}(S;H)} \|y\|_{L_{r_1}(S;H)} + \|\eta_2\|_{L_{r_2'}(S;H)} \|y\|_{L_{r_2}(S;H)} \Big) \le$$

$$\le \max_{s\in S} |\varphi'(s)| \|y\|_{C(S;V^*)} \Big(\|y\|_{L_{p_1}(S;V_1)} \text{mes}\,(S)^{1/q_1} + \|y\|_{L_{p_2}(S;V_2)} \text{mes}\,(S)^{1/q_2} \Big) +$$

$$+ 2\max_{s\in S} |\varphi(s)-1| \Big(\|\xi_1\|_{L_{q_1}(S;V_1^*)} + \|\xi_2\|_{L_{q_2}(S;V_2^*)} + \|\eta_1\|_{L_{r_1'}(S;H)} + \|\eta_2\|_{L_{r_2'}(S;H)} \Big) \times$$

$$\times \Big(\|y\|_{L_{p_1}(S;V_1)} + \|y\|_{L_{p_2}(S;V_2)} + \|y\|_{C(S;H)} \text{mes}\,(S)^{1/r_1} + \|y\|_{C(S;H)} \text{mes}\,(S)^{1/r_2} \Big).$$

Hence, due to [1, theorem 3], definition of $\|\cdot\|_X$, if we take in last inequality $\varphi(t) = \dfrac{t-a}{b-a}$ for all $t \in S$ we obtain

$$\|y\|_{C(S;H)}^2 \le C_2 \|y\|_{W_0^*}^2 + C_3 \|y\|_{W_0^*} \|y\|_{C(S;H)}, \tag{2}$$

where C_1 is the constant from inequality $\|y\|_{C(S;V^*)} \le C_1 \|y\|_{W_0^*}$ for every $y \in W_0^*$,

$$C_2 = 2 + \frac{C_1}{\min\,\{\text{mes}\,(S)^{1/p_1}, \text{mes}\,(S)^{1/p_2}\}}, \quad C_3 = 2\max\,\left\{\text{mes}\,(S)^{1/\min\{r_1, r_2\}}, 1\right\}$$

Remark that $\dfrac{1}{+\infty} = 0$, $C_2, C_3 > 0$. From (2) it obviously follows that

$$\|y\|_{C(S;H)} \le C_4 \|y\|_{W_0^*} \quad \text{for all } y \in C^1(S;V), \tag{3}$$

where $C_4 = \dfrac{C_3 + \sqrt{C_3^2 + 4C_2}}{2}$ does not depend on y.

Now let us apply [1, theorem 4]. For arbitrary $y \in W_0^*$ let $\{y_n\}_{n\ge1}$ be a sequence of elements from $C^1(S;V)$ converging to y in W_0^*. Then in virtue of relation (3) we have

$$\|y_n - y_k\|_{C(S;H)} \le C_4 \|y_n - y_k\|_{W_0^*} \to 0,$$

therefore, the sequence $\{y_n\}_{n\ge1}$ converges in $C(S;H)$ and it has only limit $\chi \in C(S;H)$ such that for a.e. $t\in S$ $\chi(t) = y(t)$. So, we have $y\in C(S;H)$ and now the embedding $W_0^* \subset C(S;H)$ is proved. If we pass to limit in (3) with $y = y_n$ as $n\to\infty$ we obtain the validity of the given estimation $\forall\, y\in W_0^*$. It proves the continuity of the embedding W^* into $C(S;H)$.

Now let us prove formula (1). For every $y,\xi \in W_0^*$ and for corresponding approximating sequences $\{y_n,\xi_n\}_{n\ge1} \subset C^1(S;V)$ we pass to the limit in (1) with $y = y_n$, $\xi = \xi_n$ as $n\to\infty$. In virtue of Lebesgue's theorem and $W_0^* \subset C(S;V^*)$ with continuous embedding formula (1) is true for every $y\in W_0^*$.

The theorem is proved.

In virtue of $W^* \subset W_0^*$ with continuous embedding and due to the latter theorem the next statement is true.

Corollary 1. $W^* \subset C(S;H)$ with continuous embedding. Moreover, for every $y,\xi\in W^*$ and $s,t\in S$ formula (1) takes place.

For every $n\ge1$ let us define the Banach space $W_n^* = \{y\in X_n^* \mid y'\in X_n\}$ with the norm

$$\|y\|_{W_n^*} = \|y\|_{X_n^*} + \|y'\|_{X_n},$$

where the derivative y' is considered in sense of scalar distributions space $\mathcal{D}^*(S;H_n)$. As far as

$$\mathcal{D}^*(S;H_n) = \mathcal{L}(\mathcal{D}(S);H_n) \subset \mathcal{L}(\mathcal{D}(S);V_\omega^*) = \mathcal{D}^*(S;V^*)$$

it is possible to consider the derivative of an element $y\in X_n^*$ in the sense of $\mathcal{D}^*(S;V^*)$. Remark that for every $n\ge1$ $W_n^* \subset W_{n+1}^* \subset W^*$.

Proposition 1. For every $y\in X^*$ and $n\ge1$ $P_n y' = (P_n y)'$, where derivative of element $x\in X^*$ is in the sense of the scalar distributions space $\mathcal{D}^*(S;V^*)$.

Remark 1. We pay our attention that in virtue of the previous assumptions the derivatives of an element $x\in X_n^*$ in the sense of $\mathcal{D}(S;V^*)$ and in the sense of $\mathcal{D}(S;H_n)$ coincide.

Proof. It is sufficient to show that for every $\varphi\in\mathcal{D}(S)$ $P_n y'(\varphi) = (P_n y)'(\varphi)$. In virtue of definition of derivative in sense of $\mathcal{D}^*(S;V^*)$ we have

$$\forall\,\varphi\in\mathcal{D}(S)\quad P_n y'(\varphi) = -P_n y(\varphi') = -P_n \int_S y(\tau)\varphi'(\tau)\,d\tau =$$

$$= -\int_S P_n y(\tau)\varphi'(\tau)d\tau = -(P_n y)(\varphi') = (P_n y)'(\varphi).$$

The proposition is proved.

Due to [1, propositions 3, 4] it follows the next

Proposition 2. For every $n \geq 1$ $W_n^* = P_n W^*$, i.e.

$$W_n^* = \{P_n y(\cdot) \mid y(\cdot) \in W^*\}.$$

Moreover, if the triple $(\{H_i\}_{i\geq 1}; V_j; H)$, $j = 1,2$ satisfies condition (γ) with $C = C_j$. Then for every $y \in W^*$ and $n \geq 1$

$$\|P_n y(\cdot)\|_{W^*} \leq \max\{C_1, C_2\} \|y(\cdot)\|_{W^*}.$$

Theorem 2. Let the triple $(\{H_i\}_{i\geq 1}; V_j; H)$, $j = 1,2$ satisfy condition (γ) with $C = C_j$. We consider bounded in X^* set $D \subset X^*$ and $E \subset X$ that is bounded in X. For every $n \geq 1$ let us consider

$$D_n := \{y_n \in X_n^* | y_n \in D \text{ and } y_n' \in P_n E\} \subset W_n^*.$$

Then

$$\|y_n\|_{W^*} \leq \|D\|_+ + C\|E\|_+ \quad \text{for all } n \geq 1 \text{ and } y_n \in D_n, \tag{4}$$

where $C = \max\{C_1, C_2\}$, $\|D\|_+ = \sup_{y\in D} \|y\|_{X^*}$ and $\|E\|_+ = \sup_{f\in E} \|f\|_X$.

Remark 2. Due to proposition 2 D_n is well-defined and $D_n \subset W_n^*$ is true.

Remark 3. A priori estimates (like (4)) appear at studying of solvability of differential–operator equations, inclusions and evolutional variational inequalities in Banach spaces with maps of w_λ-pseudomonotone type by using Faedo–Galerkin method (see [2, 3]) at boundary transition, when it is necessary obtain a priori estimates of approximate solutions y_n in X^* and of its derivatives y_n' in X.

Proof. Due to proposition 2 for every $n \geq 1$ and $y_n \in D_n$

$$\|y_n\|_{W^*} = \|y_n\|_{X^*} + \|y_n'\|_X \leq \|D\|_+ + \|P_n E\|_+ \leq \|D\|_+ + \max\{C_1, C_2\}\|E\|_+.$$

The theorem is proved.

Further, let B_0, B_1, B_2 be some Banach spaces such, that

$$B_0, B_2 \text{ are reflexive } B_0 \subset B_1 \text{ with compacting embedding} \tag{5}$$

$$B_0 \subset B_1 \subset B_2 \text{ with compacting embedding.} \tag{6}$$

Lemma 1. ([4] lemma 1.5.1, p.71) Under the assumptions (5), (6) for an arbitrary $\eta > 0$ there exists $C_\eta > 0$ such that

$$\|x\|_{B_1} \leq \eta \|x\|_{B_0} + C_\eta \|x\|_{B_2} \quad \forall\, x \in B_0.$$

Corollary 2. Let the assumptions (5), (6) for the Banach spaces B_0, B_1 and B_2 are verified, $p_1 \in [1; +\infty]$, $S = [0, T]$ and the set $K \subset L_{p_1}(S; B_0)$ such that

a) K is precompact set in $L_{p_1}(S; B_2)$;

b) K is bounded set in $L_{p_1}(S;B_0)$.

Then K is precompact set in $L_{p_1}(S;B_1)$.

Proof. Due to lemma 1 and to the norm definition in $L_{p_1}(S;B_i)$, $i=\overline{0,2}$ it follows that for an arbitrary $\eta>0$ there exists such $C_\eta>0$ that

$$\|y\|_{L_{p_1}(S;B_1)} \le 2\eta\|y\|_{L_{p_1}(S;B_0)} + 2C_\eta\|y\|_{L_{p_1}(S;B_2)} \quad \forall\, y\in L_{p_1}(S;B_0) \quad (7)$$

Let us check inequality (7), when $p_1\in[0,+\infty)$ (the case $p_1=+\infty$ is direct corollary of lemma 1):

$$\|y\|^{p_1}_{L_{p_1}(S;B_1)} = \int_S \|y(t)\|^{p_1}_{B_1}\,dt \le \int_S [\eta\|y(t)\|_{B_0} + C_\eta\|y(t)\|_{B_2}]^{p_1}\,dt \le$$

$$\le 2^{p_1-1}\left[\eta^{p_1}\int_S \|y(t)\|^{p_1}_{B_0}\,dt + C_\eta^{p_1}\int_S \|y(t)\|^{p_1}_{B_2}\,dt\right] =$$

$$= 2^{p_1-1}\left[\eta^{p_1}\|y\|^{p_1}_{L_{p_1}(S;B_0)} + C_\eta^{p_1}\|y\|^{p_1}_{L_{p_1}(S;B_2)}\right] \le$$

$$\le 2^{p_1}\left[\eta\|y\|_{L_{p_1}(S;B_0)} + C_\eta\|y\|_{L_{p_1}(S;B_2)}\right]^{p_1} \quad \forall\, y\in L_{p_1}(S;B_0).$$

The last inequality follows from

$$\frac{a^{p_1}+b^{p_1}}{2} \le (a+b)^{p_1} \le 2^{p_1-1}\left(a^{p_1}+b^{p_1}\right) \quad \forall\, a,b\ge 0.$$

Now let $\{y_n\}_{n\ge1}$ be an arbitrary sequence from K. Then by the conditions of the given statement there exists $\{y_{n_k}\}_{k\ge1}\subset\{y_n\}_{n\ge1}$ that is a Cauchy subsequence in the space $L_{p_1}(S;B_2)$. So, thanks to inequality (7) for every $k,m\ge1$

$$\|y_{n_k}-y_{n_m}\|_{L_{p_1}(S;B_1)} \le 2\eta\|y_{n_k}-y_{n_m}\|_{L_{p_1}(S;B_0)} +$$

$$+ 2C_\eta\|y_{n_k}-y_{n_m}\|_{L_{p_1}(S;B_2)} \le \eta C + 2C_\eta\|y_{n_k}-y_{n_m}\|_{L_{p_1}(S;B_2)},$$

where $C>0$ is a constant that does not depend on m,k,η. Therefore, for every $\varepsilon>0$ we can choose $\eta>0$ and $N\ge1$ such that

$$\eta C<\varepsilon/2 \quad \text{and} \quad 2C_\eta\|y_{n_k}-y_{n_m}\|_{L_{p_1}(S;B_2)}<\varepsilon/2 \quad \forall\, m,k\ge N$$

Thus,

$$\forall\,\varepsilon>0 \quad \exists\, N\ge1: \ \|y_{n_k}-y_{n_m}\|_{L_{p_1}(S;B_1)}<\varepsilon \quad \forall\, m,k\ge N.$$

This fact means, that $\{y_{n_k}\}_{k\ge1}$ converges in $L_{p_1}(S;B_1)$. The corollary is proved.

Theorem 3. Let conditions (5), (6) for B_0, B_1, B_2 are satisfied, $p_0, p_1 \in$ $\in [1;+\infty)$, S be a finite time interval and $K \subset L_{p_1}(S;B_0)$ be such, that

a) K is bounded in $L_{p_1}(S;B_0)$;

b) for every $\varepsilon > 0$ there exists such $\delta > 0$ that from $0 < h < \delta$ it results in

$$\int_S \|u(\tau) - u(\tau + h)\|_{B_2}^{p_0} d\tau < \varepsilon \quad \forall u \in K . \tag{8}$$

Then K is precompact in $L_{\min\{p_0;p_1\}}(S;B_1)$.

Furthermore, if for some $q > 1$ K is bounded in $L_q(S;B_1)$, then K is precompact in $L_p(S;B_1)$ for every $p \in [1,q)$.

Remark 4. Further we consider that every element $x \in (S \to B_i)$ is equal to $\overline{0}$ out of the interval S.

Proof. At the beginning we consider the first case. For our goal it is enough to show, that it is possible to choose a Cauchy subsequence from every sequence $\{y_n\}_{n\geq 1} \subset K$ in $L_{\min\{p_0;p_1\}}(S;B_1)$. Due to corollary 2 it is sufficient to prove this statement for $L_{\min\{p_0;p_1\}}(S;B_2)$.

For every $x \in K$ $\forall h > 0$ $\forall t \in S$ we put

$$x_h(t) := \frac{1}{h}\int_t^{t+h} x(\tau)d\tau ,$$

where the integral is regarded in the sense of Bochner integral. We point out that $\forall h > 0$ $x_h \in C(S;B_0) \subset C(S;B_2)$.

Fixing a positive number ε, we construct for a set

$$K \subset L_{p_0}(S;B_0) \subset L_{p_0}(S;B_2)$$

a final ε-web in $L_{p_0}(S;B_2)$. For $\varepsilon > 0$ we choose $\delta > 0$ from (8). Then for every fixed h ($0 < h < \delta$) we have:

$$\|x_h(t+u) - x_h(t)\|_{B_2} = \frac{1}{h}\|\int_{t+u}^{t+u+h} x(\tau)d\tau - \int_t^{t+h} x(\tau)d\tau\|_{B_2} =$$

$$= \frac{1}{h}\|\int_t^{t+h} x(\tau+u)d\tau - \int_t^{t+h} x(\tau)d\tau\|_{B_2} \leq \frac{1}{h}\int_t^{t+h} \|x(\tau+u) - x(\tau)\|_{B_2} d\tau .$$

Moreover, from the Hölder inequality we obtain

$$\frac{1}{h}\int_t^{t+h} \|x(\tau+u) - x(\tau)\|_{B_2} d\tau \leq \left(\frac{1}{h}\right)^{\frac{1}{p_0}} \left(\int_t^{t+h} \|x(\tau+u) - x(\tau)\|_{B_2}^{p_0} d\tau\right)^{\frac{1}{p_0}} \leq$$

$$\le \left(\frac{1}{h}\right)^{\frac{1}{p_0}} \left(\int_0^T \|x(\tau+u) - x(\tau)\|_{B_2}^{p_0} d\tau\right)^{\frac{1}{p_0}} < \left(\frac{\varepsilon}{h}\right)^{\frac{1}{p_0}} \quad \forall x \in K, \ \forall\, 0 < u < \delta, \ \forall t \in S\,.$$

Therefore the family of functions $\{x_h\}_{x \in K}$ is equicontinuous.

Since $\forall x \in K \quad \forall t \in S$ it results in

$$\|x_h(t)\|_{B_2} = \frac{1}{h}\|\int_t^{t+h} x(\tau)d\tau\|_{B_2} \le \frac{1}{h}\int_t^{t+h} \|x(\tau)\|_{B_2} d\tau \le$$

$$\le \left(\frac{1}{h}\right)^{\frac{1}{p_1}} \left(\int_t^{t+h} \|x(\tau)\|_{B_2}^{p_1} d\tau\right)^{\frac{1}{p_1}} \le \left(\frac{1}{h}\right)^{\frac{1}{p_1}} \left(\int_0^T \|x(\tau)\|_{B_2}^{p_1} d\tau\right)^{\frac{1}{p_1}} \le \left(\frac{C}{h}\right)^{\frac{1}{p_1}},$$

the family of functions $\{x_h\}_{x\in K}$ is uniformly bounded, because of the constant $C \ge 0$ does not depend on $x \in K$. Hence, $\forall h : 0 < h < \delta$ the family of functions $\{x_h\}_{x\in K}$ is precompact in $C(S; B_2)$, so in $L_{\min\{p_0,p_1\}}(S; B_2)$ too.

On the other hand, $\forall\, 0 < h < \delta$, $\forall x \in K$, $\forall t \in S$

$$\|x(t) - x_h(t)\|_{B_2} \le \frac{1}{h}\int_t^{t+h} \|x(t) - x(\tau)\|_{B_2} d\tau \le$$

$$\le \frac{1}{h}\int_0^h \|x(t) - x(t+\tau)\|_{B_2} d\tau \le \left(\frac{1}{h}\right)^{\frac{1}{p_0}} \left(\int_0^h \|x(t) - x(t+\tau)\|_{B_2}^{p_0} d\tau\right)^{\frac{1}{p_0}}.$$

From here, taking into account inequality (8) we receive:

$$\left(\int_0^T \|x(t) - x_h(t)\|_{B_2}^{p_0} dt\right)^{\frac{1}{p_0}} \le \left(\int_0^T \frac{1}{h}\int_0^h \|x(t) - x(t+\tau)\|_{B_2}^{p_0} d\tau dt\right)^{\frac{1}{p_0}} =$$

$$= \left(\frac{1}{h}\int_0^h\int_0^T \|x(t) - x(t+\tau)\|_{B_2}^{p_0} dt d\tau\right)^{\frac{1}{p_0}} < \left(\frac{1}{h}\int_0^h \varepsilon d\tau\right)^{\frac{1}{p_0}} = \varepsilon^{\frac{1}{p_0}}.$$

Hence, by virtue of the precompactness of system $\{x_h\}_{x\in K}$ in $L_{\min\{p_0,p_1\}}(S; B_2)$ $\forall\, 0 < h < \delta$ we have that K is a precompact set in $L_{\min\{p_0,p_1\}}(S; B_2)$.

Let us consider the second case. Assume that for some $q > 1$ the set K is bounded in $L_q(S; B_1)$. Similarly to the previous case, it is enough to show that for every $p \in [1; q)$ and $\{y_n\}_{n\ge1} \subset K$ there exists a subsequence $\{y_{n_k}\}_{k\ge1} \subset \subset \{y_n\}_{n\ge1}$ and $y \in L_p(S; B_1)$ so that

$$y_{n_k} \to y \quad \text{in} \quad L_p(S; B_1) \quad \text{as} \quad k \to \infty\,.$$

Because of $y_n \to y$ in $L_{\min\{p_0,p_1\}}(S;B_1)$, up to a subsequence, as $n \to \infty$, we have $\exists \{y_{n_k}\}_{k\geq 1} \subset \{y_n\}_{n\geq 1}$ such that $\lambda(B_{n_k}) \to 0$ as $k \to \infty$, where $B_n :=$ $:= \{t \in S \,|\, \|y_n(t) - y(t)\|_{B_1} \geq 1\}$ for every $n \geq 1$, λ is the Lebesgue measure on S. Then for every $k \geq 1$

$$\int_S \|y_{n_k}(s) - y(s)\|_{B_1}^p ds = \int_{A_{n_k}} \|y_{n_k}(s) - y(s)\|_{B_1}^p ds +$$

$$+ \int_{B_{n_k}} \|y_{n_k}(s) - y(s)\|_{B_1}^p ds \leq \int_{A_{n_k}} \|y_{n_k}(s) - y(s)\|_{B_1}^p ds +$$

$$+ \left(\int_S \|y_{n_k}(s) - y(s)\|_{B_1}^q ds \right)^{\frac{p}{q}} \left(\lambda(B_{n_k}) \right)^{\frac{q-p}{q}} =: I_{n_k} + J_{n_k},$$

where $A_n = S \setminus B_n$ for every $n \geq 1$.

It is clear that $J_{n_k} \to 0$ as $k \to \infty$. Let us consider I_{n_k}. Since $\{y_{n_k}\}_{k\geq 1}$ is precompact in $L_{\min\{p_0,p_1\}}(S;B_1)$, there exists such $\{y_{m_k}\}_{k\geq 1} \subset \{y_{n_k}\}_{k\geq 1}$ that $y_{m_k}(t) \to y(t)$ in B_1 as $k \to \infty$ almost everywhere in S. Setting

$$\forall k \geq 1, \quad \forall t \in S \quad \varphi_{m_k}(t) := \begin{cases} \|y_{m_k}(t) - y(t)\|_{B_1}^p, & t \in A_n, \\ 0, & \text{otherwice}, \end{cases}$$

using definition of A_{m_k}, sequence $\{\varphi_{m_k}\}_{k\geq 1}$ satisfies the conditions of the Lebesgue theorem with the integrable majorant $\phi \equiv 1$. So $\varphi_{m_k} \to \bar{0}$ in $L_1(S)$ as $k \to \infty$. Thus, within to a subsequence, $y_n \to y$ in $L_q(S;B_1)$.

The theorem is proved.

Let Banach spaces B_0, B_1, B_2 satisfy all assumptions (5), (6), $p_0, p_1 \in [1;+\infty)$ be arbitrary numbers. We consider the set with the natural operations

$$W = \{v \in L_{p_0}(S;B_0) | v' \in L_{p_1}(S;B_2)\},$$

where the derivative v' of an element $v \in L_{p_0}(S;B_0)$ is considered in the sense of the scalar distribution space $\mathcal{D}(S;B_2)$. It is clear, that

$$W \subset L_{p_0}(S;B_0).$$

Theorem 4. The set W with the natural operations and the graph norm

$$\|v\|_W = \|v\|_{L_{p_0}(S;B_0)} + \|v'\|_{L_{p_1}(S;B_2)}$$

is a Banach space.

Proof. The executing of the norm properties for $\|\cdot\|_W$ immediately follows from its definition. Now we consider the completeness of W referring to just defined norm. Let $\{v_n\}_{n\geq 1}$ be a Cauchy sequence in W. Hence, due to the completeness of $L_{p_0}(S;B_0)$ and $L_{p_1}(S;B_2)$ it follows that for some $y\in L_{p_0}(S;B_0)$ and $v\in L_{p_1}(S;B_2)$

$$y_n \to y \text{ in } L_{p_0}(S;B_0) \text{ and } y_n' \to v \text{ in } L_{p_1}(S;B_2) \text{ as } n\to +\infty .$$

Due to [5, lemma IV.1.10] and in virtue of continuous dependence of the derivative by the distribution in $\mathcal{D}^*(S;B_2)$ (see [5, p. 169) it follows, that $y' = v \in L_{p_1}(S;B_2)$.

The theorem is proved.

Theorem 5. Under conditions (5), (6) $W\subset C(S;B_2)$ with the continuous embedding.

Proof. For a fixed $y\in W$ let us show that $y\in C(S;B_2)$. Let us put

$$\xi(t) = \int_{t_0}^{t} y'(\tau)d\tau \quad \forall\, t_0, t\in S .$$

The integral is well-defined because $y'\in L_1(S;B_2)$. On the other hand, from the inequality [5, p. 153]

$$\|\xi(t)-\xi(s)\|_{B_2} \leq \int_t^s \|y'(\tau)\|_{B_2} d\tau \quad \forall\, s\geq t,\ s\in S$$

it follows that $\xi\in C(S;B_2)$. Due to [5] (lemma IV.1.8) $\xi' = y'$, so from [5] (lemma IV.1.9) it follows that

$$y(t) = \xi(t) + z \quad \text{for a.e.} \quad t\in S .$$

for some fixed $z\in B_2$.

Thus the function y also lies in $C(S;B_2)$.

In virtue of the continuous embedding of $L_{p_1}(S;B_2)$ in $L_1(S;B_2)$ we have that for some constant $k>0$, which does not depend on y,

$$\|\xi(t)\|_{B_2} \leq \int_S \|y'(\tau)\|_{B_2} d\tau \leq k\|y'\|_{L_{p_1}(S;B_2)} \quad \forall\, t\in S .$$

From here, due to the continuous embedding $B_0\subset B_2$, we have

$$\|z\|_{B_2} (\mathrm{mes}(S))^{1/p_1} = \left(\int_S \|z\|_{B_2}^{p_1} ds \right)^{1/p_1} = \|y-\xi\|_{L_{p_1}(S;B_2)} \leq$$

$$\leq k_1 \left(\|y\|_{L_{p_1}(S;B_2)} + \|\xi\|_{C(S;B_2)} \right) \leq k_2 \left(\|y\|_{L_{p_0}(S;B_0)} + \|y'\|_{L_{p_1}(S;B_2)} \right),$$

where $\text{mes}(S)$ is the "length"' (the measure) of S, $k_2 > 0$ is a constant that does not depend on $y \in W$. Therefore, from the last two relations there exists $k_3 \geq 0$ such that

$$\|y\|_{C(S;B_2)} \leq k_3 \|y\|_W \quad \forall\, y \in W .$$

The theorem is proved.

The next result represents a generalization of the compactness lemma [4, theorem 1.5.1, p. 70] into the case $p_0, p_1 \in [1;+\infty)$.

Theorem 6. Under conditions (5), (6), for all $p_0, p_1 \in [1;+\infty)$ the Banach space W is compactly embedded in $L_{p_0}(S;B_1)$.

Proof. At the beginning we prove the compact embedding of W in $L_1(S;B_2)$.

For every $y \in W$ and $h \in \mathbb{R}$ let us take

$$y_h(t) = \begin{cases} y(t+h), & \text{if } \ t+h \in S, \\ \bar{0}, & \text{otherwice.} \end{cases}$$

In virtue of theorem 5 the given definition is correct.

Lemma 2. For every $y \in W$ and $h \in \mathbb{R}$

$$\|y - y_h\|_{L_1(S;B_2)} \leq h \|y'\|_{L_1(S;B_2)} . \tag{9}$$

Proof. Let $y \in W$ be fixed. Then

$$\|y - y_h\|_{L_1(S;B_2)} = \int_S \|y(t+h) - y(t)\|_{B_2} dt = \int_S \| \int_t^{t+h} y'(\tau) d\tau \|_{B_2} dt .$$

Let us put $g_y(t) = \int_t^{t+h} y'(\tau) d\tau = y(t+h) - y(t) \quad \forall\, t \in S,\ i = 1,2$. Due to theorem 5 the element $g_y \in C(S;B_2)$. So, as S is a compact set, we have that $g_y \in L_1(S;B_2)$. Therefore, due to proposition [6, p.191] with $X = L_1(S;B_2)$ and to [1, theorem 2] it follows the existence of $h_y \in L_\infty(S;B_2^*) \equiv X^*$ such that

$$\int_S \|g_y(t)\|_{B_2} dt = \int_S \langle h_y(t), g_y(t) \rangle_{B_2} dt \ \text{ and } \ \|h_y\|_{L_\infty(S;B_2^*)} = 1$$

Hence,

$$\int_S \left\| \int_t^{t+h} y'(\tau) d\tau \right\|_{B_2} dt = \int_S \left\| g_y(t) \right\|_{B_2} dt = \int_S \left\langle h_y(t), g_y(t) \right\rangle_{B_2} dt =$$

$$= \int_S \left\langle h_y(t), \int_t^{t+h} y'(\tau) d\tau \right\rangle_{B_2} dt = \int_S \int_t^{t+h} \left\langle h_y(t), y'(\tau) \right\rangle_{B_2} d\tau\, dt =$$

$$= \int_S \int_{\tau-h}^{\tau} \left\langle h_y(t), y'(\tau) \right\rangle_{B_2} dt d\tau = \int_S \left\langle \int_{\tau-h}^{\tau} h_y(t) dt, y'(\tau) \right\rangle_{B_2} d\tau \le$$

$$\le \operatorname{esssup}_{t \in S} \|h_y(t)\|_{B_2^*} h \int_S \|y'(\tau)\|_{B_2} d\tau \le h \|y'\|_{L_1(S;B_2)} .$$

So, we have obtained necessary estimation (9).

The lemma is proved.

Let us continue the proof of the given theorem. Let $K \subset W$ be an arbitrary bounded set. Then for some $C > 0$

$$\|y\|_{L_{p_0}(S;B_0)} \le C, \quad \|y'\|_{L_{p_1}(S;B_2)} \le C \quad \forall\, y \in K . \tag{10}$$

In order to prove the precompactness of K in $L_1(S;B_1)$ let us apply theorem 4 with $B_0 = B_0$, $B_1 = B_1$, $B_2 = B_2$, $p_0 = 1$, $p_1 = p_1$. Due to estimates (9) and (10) the all conditions of the given theorem hold. So, the set K is precompact in $L_1(S;B_1)$ and hence in $L_1(S;B_2)$. In virtue of theorem 5 and the Lebesgue theorem it follows that the set K is precompact in $L_{p_0}(S;B_0)$. Hence, due to corollary 2 we obtain the necessary statement.

The theorem is proved.

Proposition 3. Let Banach spaces B_0, B_1, B_2 satisfy conditions (5), (6), $p_0, p_1 \in [1;+\infty)$, $\{u_h\}_{h\in I} \subset L_{p_1}(S;B_0)$, where $I = (0,\delta) \subset \mathbb{R}_+$, $S = [a,b]$ such that

a) $\{u_h\}_{h \in I}$ is bounded in $L_{p_1}(S;B_0)$;

b) there exists such $c : I \to \mathbb{R}_+$ that $\varliminf_{n\to\infty} c\left(\frac{b-a}{2^n}\right) = 0$ and

$$\forall\, h \in I \quad \int_S \|u_h(t) - u_h(t+h)\|_{B_2}^{p_0} dt \le c(h) h^{p_0} .$$

Then there exists $\{h_n\}_{n\ge1} \subset I$ ($h_n \searrow 0+$ as $n \to \infty$) so that $\{u_{h_n}\}_{n\ge1}$ converges in $L_{\min\{p_0,p_1\}}(S;B_1)$.

Remark 5. We assume $u_h(t) = \bar{0}$ when $t > b$.

Remark 6. Without loss of generality let us consider $S = [0,1]$.

Proof. At first we prove this statement for $L_{p_0}(S;B_2)$. In virtue of Minkowski inequality for every $h = \frac{1}{2^N} \in I$ and $k \ge 1$

$$\left(\int_0^1 \|u_h(t) - u_{\frac{h}{2^k}}(t)\|_{B_2}^{p_0} dt \right)^{\frac{1}{p_0}} \le \left(\int_0^1 \|u_h(t) - u_h(t+h)\|_{B_2}^{p_0} dt \right)^{\frac{1}{p_0}} +$$

$$+\left(\int_0^1 \|u_h(t+h)-u_{\frac{h}{2^k}}(t+h)\|_{B_2}^{p_0}\,dt\right)^{\frac{1}{p_0}}+\left(\int_0^1 \|u_{\frac{h}{2^k}}(t+h)-u_{\frac{h}{2^k}}(t)\|_{B_2}^{p_0}\,dt\right)^{\frac{1}{p_0}}\le$$

$$\le c^{\frac{1}{p_0}}(h)h+\left(\int_h^1 \|u_h(t)-u_{\frac{h}{2^k}}(t)\|_{B_2}^{p_0}\,dt\right)^{\frac{1}{p_0}}+\sum_{i=0}^{2^k-1}\left(\int_0^1\left\|u_{\frac{h}{2^k}}\left(t+\frac{i+1}{2^k}h\right)-\right.\right.$$

$$\left.\left.-u_{\frac{h}{2^k}}\left(t+\frac{i}{2^k}h\right)\right\|_{B_2}^{p_0}dt\right)^{\frac{1}{p_0}}\le c^{\frac{1}{p_0}}(h)h+2^k\frac{h}{2^k}c^{\frac{1}{p_0}}(h/2^k)+$$

$$+\left(\int_h^1 \|u_h(t)-u_{\frac{h}{2^k}}(t)\|_{B_2}^{p_0}\,dt\right)^{\frac{1}{p_0}}\le h\left(c^{\frac{1}{p_0}}(h)+c^{\frac{1}{p_0}}(h/2^k)\right)+$$

$$+\left(\int_h^1 \|u_h(t)-u_h(t+h)\|_{B_2}^{p_0}\,dt\right)^{\frac{1}{p_0}}+\left(\int_h^1 \|u_h(t+h)-u_{\frac{h}{2^k}}(t+h)\|_{B_2}^{p_0}\,dt\right)^{\frac{1}{p_0}}+$$

$$+\left(\int_h^1 \|u_{\frac{h}{2^k}}(t+h)-u_{\frac{h}{2^k}}(t)\|_{B_2}^{p_0}\,dt\right)^{\frac{1}{p_0}}\le\ldots\le 2h\left(c^{\frac{1}{p_0}}(h)+c^{\frac{1}{p_0}}(h/2^k)\right)+$$

$$+\left(\int_{2h}^1 \|u_h(t)-u_h(t+h)\|_{B_2}^{p_0}\,dt\right)^{\frac{1}{p_0}}\le\ldots\le 2^N h\left(c^{\frac{1}{p_0}}(h)+c^{\frac{1}{p_0}}(h/2^k)\right)=$$

$$=c^{\frac{1}{p_0}}(h)+c^{\frac{1}{p_0}}(h/2^k).$$

So, for every $N\ge 1$ and $k\ge 1$ it results in

$$\left(\int_0^1 \|u_{1/2^N}(t)-u_{1/2^{N+k}}(t)\|_{B_2}^{p_0}\,dt\right)^{\frac{1}{p_0}}\le c^{\frac{1}{p_0}}\left(\frac{1}{2^N}\right)+c^{\frac{1}{p_0}}\left(\frac{1}{2^{N+k}}\right).$$

In virtue of assumption b) we can choose $\{h_n\}_{n\ge1}\subset\left\{\frac{1}{2^m}\right\}_{m\ge1}\bigcap I$ such that $c(h_n)\to 0$ as $n\to\infty$. So, the sequence $\{u_{h_n}\}_{n\ge1}$ is fundamental in $L_{p_0}(S;B_2)$. Because of $B_0\subset B_1$ with compact embedding, the sequence $\{u_{h_n}\}_{n\ge1}$ is bounded in $L_{\min\{p_0,p_1\}}(S;B_0)$; due to corollary 2 it follows that $\{u_{h_n}\}_{n\ge1}$ is fundamental in $L_{\min\{p_0,p_1\}}(S;B_1)$.

The proposition is proved.

Now we combine all results to obtain the necessary a priori estimate.

Theorem 7. Let all conditions of theorem 2 are satisfied and $V \subset H$ with compact embedding. Then (4) be true and the set

$$\bigcup_{n\geq 1} D_n \text{ is bounded in } C(S;H) \text{ and precompact in } L_p(S;H)$$

for every $p \geq 1$.

Proof. Estimation (4) follows from theorem 2. Now we use compactness theorem 6 with $B_0 = V$, $B_1 = H$, $B_2 = V^*$, $p_0 = 1$, $p_1 = 1$. Remark that $X^* \subset L_1(S;V)$ and $X \subset L_1(S;V^*)$ with continuous embedding. Hence, the set

$$\bigcup_{n\geq 1} D_n \text{ is precompact in } L_1(S;H).$$

In virtue of (4) and theorem 1 on continuous embedding of W^* in $C(S;H)$, it follows that the set

$$\bigcup_{n\geq 1} D_n \text{ is bounded in } C(S;H).$$

Further, by using standard conclusions and the Lebesgue theorem we obtain the necessary statement.

The theorem is proved.

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