UDC 517.9

ON SOME TOPOLOGICAL PROPERTIES FOR SPECIAL CLASSES OF BANACH SPACES. PART 1

P. KASYANOV, V. MEL'NIK

We consider some classes of infinite-dimensional Banach spaces with integrable derivatives. A compactness lemma for nonreflexive spaces is obtained. However some main topological properties for the given spaces are obtained.

Method of monotony and method of compactness represent fundamental approaches to study nonlinear differential-operator equations, evolutionary inclusions and variational inequalities in Banach spaces. The general idea is the following: using the corresponding approximation scheme the approximate solutions of a problem are constructed, for them some approaching a priori estimations are established, at last they prove the existence of sequence of approximate solutions, that converges to the exact solution of problem. In many cases the aim is obtained by using both a method of compactness and a method of monotonicity.

In the present paper we obtain a new of compact embedding theorems for Banach spaces, suggested by researches about differential-operational inclusions in function spaces. Moreover, we introduce some constructions to prove the convergence of Faedo–Galerkin method for evolution variation inequalities with $w_\lambda-$ pseudomonotone maps [1–5].

In the following referring to Banach spaces X, Y, when we write

$$X \subset Y$$

we mean the embedding in the set-theory sense and in the topological sense.

For $n \ge 2$ let $\{X_i\}_{i=1}^n$ be some family of Banach spaces.

Definition 1. The interpolation family is refers a family of Banach spaces $\{X_i\}_{i=1}^n$ such that for some locally convex linear topological space (LTS) Y we have

$$X_i \subset Y \quad \text{for all} \quad i = \overline{1,n}.$$

As $n = 2$ the interpolation family is called the interpolation pair.

Further let $\{X_i\}_{i=1}^n$ be some interpolation family. On the analogy of ([6], p. 23), in the linear variety $X = \cap_{i=1}^n X_i$ we consider the norm

$$\|x\|_X := \sum_{i=1}^n \|x\|_{X_i} \quad \forall\, x \in X\,, \tag{1}$$

where $\|\cdot\|_{X_i}$ is the norm in X_i.

Proposition 1. Let $\{X, Y, Z\}$ be an interpolation family. Then

$$X \cap (Y \cap Z) = (X \cap Y) \cap Z = X \cap Y \cap Z, \quad X \cap Y = Y \cap X$$

both in the sense of equality of sets and in the sense of equality of norms.

We also consider the linear space

$$Z := \sum_{i=1}^{n} X_i = \left\{ \sum_{i=1}^{n} x_i \,\middle|\, x_i \in X_i ,\ i = \overline{1,n} \right\}$$

with the norm

$$\|z\|_Z := \inf \left\{ \max_{i=\overline{1,n}} \|x_i\|_{X_i} \,\middle|\, x_i \in X_i ,\ \sum_{i=1}^{n} x_i = z \right\} \quad \forall z \in Z. \tag{2}$$

Proposition 2. Let $\{X_i\}_{i=1}^{n}$ be an interpolation family. Then $X = \cap_{i=1}^{n} X_i$ and $Z = \sum_{i=1}^{n} X_i$ are Banach spaces and it results in

$$X \subset X_i \subset Z \quad \text{for all} \quad i = \overline{1,n}. \tag{3}$$

Proof. Since X is a linear space, from properties of $\|\cdot\|_{X_i}$ and from the definition of $\|\cdot\|_X$ on X it follows that $\|\cdot\|_X$ is the norm on X.

Let us prove the completeness of X. From the definition of $\|\cdot\|_X$ on X it follows that every Cauchy sequence $\{x_n\}_{n\geq 1}$ in X is fundamental, so it converges in X_i and in Y $\forall\, i = \overline{1,n}$, where Y is the LTS in the definition 1. Hence, due to $\{X_i\}_{i=1}^{n}$ is the interpolation family and to the uniqueness of the limit of a sequence $\{x_n\}_{n\geq 1}$ in LTS Y it follows that for some $x \in X$ and for all $i = \overline{1,n}$

$$x_n \to x \quad in \quad X_i \quad \text{as} \quad n \to \infty.$$

So, $x_n \to x$ in X as $n \to \infty$.

Now let us check that $\|\cdot\|_Z$ is the norm on Z.

If $\|z\|_Z = 0$, then thanks to (2) for each $m \geq 1$ there exists $x_{mi} \in X_i$ ($i = \overline{1,n}$) such that

$$z = \sum_{i=1}^{n} x_{mi}, \quad \|x_{mi}\|_{X_i} < \frac{1}{n}.$$

For every $i = \overline{1,n}$ the sequence x_{mi} tends to $\bar{0}$ in X_i, and so in Y too. Thus $\sum_{i=1}^{n} x_{mi} \to \bar{0}$ in Y as $m \to +\infty$ and $z = \bar{0}$. On the other hand, let $z = \bar{0}$. Then $\|z\|_Z \leq \max_{i=\overline{1,n}} \|0\|_{X_i} = 0$.

The another norm properties for $\|\cdot\|_Z$ follow from the properties of $\inf$, $\max$ and norms $\|\cdot\|_{X_i}$, $i = \overline{1,n}$.

Let us check Z under the above norm is complete space. Let $\{z_m\}_{m\geq 1}$ be a Cauchy sequence in Z. It contains a subsequence $\{z_{m_k}\}_{k\geq 1}$ with the property

$$\left\|z_{m_k} - z_{m_{k-1}}\right\|_Z < 2^{-k} \quad \text{for} \quad k \geq 2.$$

From (2) for every $k \geq 2$ there exists

$$z_{m_k} - z_{m_{k-1}} = \sum_{j=1}^{n} u_{kj},$$

where $u_{kj} \in X_j$, $\left\|u_{kj}\right\|_X < 2^{1-k}$ for each $j = \overline{1,n}$ and $k \geq 2$. Further,

$$z_{m_1} = \sum_{j=1}^{n} u_{1j},\ u_{1j} \in X_j,\ j = \overline{1,n}.$$

For every $k \geq 1$ let us put

$$x_{kj} = \sum_{i=1}^{k} u_{ij}, \quad j = \overline{1,n}.$$

Hence

$$z_{m_k} = \sum_{j=1}^{n} x_{kj} \quad \forall k \geq 1.$$

For all $j = \overline{1,n}$ the sequence x_{kj} converges in X_j (according to its construction) to some $x_j \in X_j$. Let us set $z = \sum_{j=1}^{n} x_j$. Then we have

$$\left\|z - z_{m_k}\right\|_Z \leq \max_{j=\overline{1,n}} \left\|x_j - x_{kj}\right\|_{X_j} \quad \forall k \geq 1.$$

From here it follows that z_{m_k} converges to z in Z as $k \to +\infty$. From the estimation

$$\left\|z - z_m\right\|_Z \leq \left\|z - z_{m_k}\right\|_Z + \left\|z_{m_k} - z_m\right\|_Z$$

and taking into account that the sequence $\{z_m\}_{m\geq 1}$ is fundamental we obtain

$$\lim_{m\to\infty} \left\|z - z_m\right\|_Z = 0.$$

The embedding (3) follows from the definition of Banach spaces $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$.

Remark 1. ([6], p. 24). Let Banach spaces X and Y satisfy the following conditions

$$X \subset Y, \qquad X \text{ is dense in } Y,$$
$$\|x\|_Y \leq \gamma \|x\|_X \quad \forall x \in X, \quad \gamma = \text{const}.$$

Then

$$Y^* \subset X^*, \quad \|f\|_{X^*} \leq \gamma \|f\|_{Y^*} \quad \forall f \in Y^*.$$

Moreover, if X is reflexive, then Y^* is dense in X^*.

Let $\{X_i\}_{i=1}^n$ be an interpolation family such that the space $X := \bigcap_{i=1}^n X_i$ with the norm (1) is dense in X_i for all $i=\overline{1,n}$. Due to remark 4 the space X_i^* may be considered as subspace of X^*. Thus we can construct $\sum_{i=1}^n X_i^*$ and

$$\sum_{i=1}^n X_i^* \subset \left(\bigcap_{i=1}^n X_i\right)^*. \tag{4}$$

Under the given assumptions X is dense in $Z := \sum_{i=1}^n X_i$ for every $i=\overline{1,n}$. So X_i is dense in Z too. Thanks to remark 1 we can consider space Z^* as a subspace of X_i^* for all $i=\overline{1,n}$, and also as a subspace of $\bigcap_{i=1}^n X_i^*$, i.e.

$$\left(\sum_{i=1}^n X_i\right)^* \subset \bigcap_{i=1}^n X_i^*. \tag{5}$$

Theorem 1. Let $\{X_i\}_{i=1}^n$ be an interpolation family such that the space $X := \bigcap_{i=1}^n X_i$ with the norm (1) is dense in X_i for all $i=\overline{1,n}$. Then

$$\sum_{i=1}^n X_i^* = \left(\bigcap_{i=1}^n X_i\right)^* \text{ and } \left(\sum_{i=1}^n X_i\right)^* = \bigcap_{i=1}^n X_i^*$$

both in the sense of sets equality and in the sense of the equality of norms.

Proof. We consider the space $\mathcal{X} := \prod_{i=1}^n X_i$ with the norm

$$\|\{x_1, x_2, ..., x_n\}\|_{\mathcal{X}} = \sum_{i=1}^n \|x_i\|_{X_i} \quad \forall\, x = \{x_1, x_2, ..., x_n\} \in \mathcal{X};$$

let $\mathcal{L}$ be the subspace of $\mathcal{X}$ defined by

$$\mathcal{L} = \{\{x, x, ..., x\} | x \in X\}.$$

For a fixed $f \in X^*$ let us set

$$u(\{x, x, ..., x\}) = f(x) \quad \forall\, x \in X.$$

Hence u is a linear functional on $\mathcal{L}$ with the norm $\|u\|_* = \|f\|_{X^*}$. By Hahn–Banach theorem for the functional u there exists a linear functional v defined on $\mathcal{X}$ such that

$$\|v\| = \|u\|_* = \|f\|_{X^*}.$$

For every $i=\overline{1,n}$ we set

$$g_i(x) = v(\{\bar{0}, ..., \bar{0}, x_i, \bar{0}, ..., \bar{0}\}) \quad \forall\, x_i \in X_i.$$

Hence it is clear that $g_i \in X_i^*$ for all $i=\overline{1,n}$ and

$$\max_{i=\overline{1,n}} \|g_i\|_{X_i^*} \le \|v\|_{Z^*} = \|f\|_{X^*}.$$

By the construction,

$$f(x) = \sum_{i=1}^{n} g_i(x) \quad \forall x \in X,$$

i.e. $f = \sum_{i=1}^{n} g_i \in \sum_{i=1}^{n} X_i^*$. Thus it follows

$$\|f\|_{\sum_{i=1}^{n} X_i^*} \le \max_{i=\overline{1,n}} \|g_i\|_{X_i^*} \le \|f\|_{X^*}.$$

On the other hand

$$\|f\|_{X^*} = \sup_{\sum_{i=1}^{n} \|f\|_{X_i} = 1} f(x) \le$$

$$\le \sup_{\sum_{i=1}^{n} \|x\|_{X_i} = 1} \inf \left\{ \sum_{i=1}^{n} \|g_i\|_{X_i^*} \|x\|_{X_i} \;\middle|\; g_i \in X_i^*,\ \sum_{i=1}^{n} g_i = f \right\} \le$$

$$\le \inf \left\{ \max_{i=\overline{1,n}} \|g_i\|_{X_i^*} \;\middle|\; g_i \in X_i^*, \sum_{i=1}^{n} g_i = f \right\} = \|f\|_{\sum_{i=1}^{n} X_i^*}.$$

The latest inequalities and (4) prove the first part of the theorem.

Let us prove the remaining part.

Lemma 1. Let $f \in \bigcap_{i=1}^{n} X_i^*$. Then for every $k = \overline{2,n}$ and $x_i, y_i \in X_i$ ($i = \overline{1,k}$) such that $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i =: x$ we have

$$\sum_{i=1}^{k} f(x_i) = \sum_{i=1}^{k} f(y_i) =: f(x). \tag{6}$$

Proof. We prove this statement arguing by induction.

Let $x_i, y_i \in X_i$ ($i = 1,2$) such that $x_1 + x_2 = y_1 + y_2 =: x$. Then $x_1 - y_1 = = y_2 - x_2 \in X_1 \bigcap X_2$ and

$$f(x_1) - f(y_1) = f(x_1 - y_1) = f(y_2 - x_2) = f(y_2) - f(x_2).$$

From the last the necessary statement is follows.

Now we assume that for some $k = \overline{2, n-1}$ and for arbitrary $x_i, y_i \in X_i$ ($i = \overline{1,k}$) such that $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i =: x$ equality (6) is valid.

Let $x_i, y_i \in X_i$ ($i = \overline{1, k+1}$) such that $\sum_{i=1}^{k+1} x_i = \sum_{i=1}^{k+1} y_i =: x$. Thus

$$x_{k+1} - y_{k+1} = \sum_{i=1}^{k} (y_i - x_i) \in \left(\sum_{i=1}^{k} X_i \right) \bigcap X_{k+1},$$

and so, by the induction assumption, we obtain

$$f(x_{k+1})-f(y_{k+1})=f(x_{k+1}-y_{k+1})=f\left(\sum_{i=1}^{k}(y_i-x_i)\right)=\sum_{i=1}^{k}(f(y_i)-f(x_i))$$

and the lemma follows.

According to lemma 1 let us continues any fixed functional $f\in\bigcap_{i=1}^{n}X_i^*$ to some functional on Z in such way:

for $z=\sum_{i=1}^{n}x_i$, where $x_i\in X_i\quad\forall i=\overline{1,n}$,

$$f(z)=\sum_{i=1}^{n}f(x_i).$$

From relation (6) it follows that the given definition is correct and does not depend on the representation of z as $\sum_{i=1}^{n}x_i$. Since

$$f(z)\le\inf\left\{\sum_{i=1}^{n}\|f\|_{X_i^*}\|x_i\|_{X_i}\,\middle|\,x_i\in X_i,\sum_{i=1}^{n}x_i=z\right\}\le\left(\sum_{i=1}^{n}\|f\|_{X_i^*}\right)\|z\|_Z,$$

then $f\in Z^*$ and $\|f\|_{Z^*}\le\|f\|_{\cap_{i=1}^{n}X_i^*}$. Taking into account (5) we have $Z^*=$ $=\bigcap_{i=1}^{n}X_i^*$ as equality of the sets. In order to prove the equality of norms it is sufficient to show the inequality $\|f\|_{\cap_{i=1}^{n}X_i^*}\le\|f\|_{Z^*}$. For every $\varepsilon>0$ there exists $x_i\in X_i$ such that

$$\|f\|_{X_i^*}\le f(x_i)+\varepsilon/n,\quad \|x_i\|_{X_i}=1.$$

Hence

$$\|f\|_{\cap_{i=1}^{n}X_i^*}=\sum_{i=1}^{n}\|f\|_{X_i^*}\le f\left(\sum_{i=1}^{n}x_i\right)+\varepsilon\le\|f\|_{Z^*}\left\|\sum_{i=1}^{n}x_i\right\|_Z+\varepsilon\le$$

$$\le\|f\|_{Z^*}\max_{i=\overline{1,n}}\|f\|_{X_i}+\varepsilon=\|f\|_{Z^*}+\varepsilon$$

and from here the delivered conclusion follows.

Now let Y be some Banach space, Y^* its topological conjugated space, S be some compact time interval. We consider the classes of functions defined on S and imagines in Y (or in Y^*).

The set $L_p(S;Y)$ of all measured by Bochner functions [6] as $1\le p\le+\infty$ with the natural linear operations with the norm

$$\|y\|_{L_p(S;Y)}=\left(\int_S\|y(t)\|_Y^p\,dt\right)^{1/p}$$

is a Banach space. As $p=+\infty$ $L_\infty(S;Y)$ with the norm

$$\|y\|_{L_\infty(S;Y)}=\operatorname{vrai\,max}_{t\in S}\|y(t)\|_Y$$

is a Banach space.

The next theorem shows that under the assumption of reflexivity or separability of Y the conjugated to $L_p(S;Y)$, $1 \le p < +\infty$, space $(L_p(S;Y))^*$ may be identify with $L_q(S;Y^*)$, where q is such that $p^{-1} + q^{-1} = 1$.

Theorem 2. If the space Y is reflexive and $1 \le p < +\infty$, then each element $f \in (L_p(S;Y))^*$ has the unique representation

$$f(y) = \int_S \langle \xi(t), y(t) \rangle_Y \, dt \quad \text{for every} \quad y \in L_p(S;Y)$$

with the function $\xi \in L_q(S;Y^*)$, $p^{-1} + q^{-1} = 1$. The correspondence $f \to \xi$, with $f \in (L_p(S;Y))^*$ is linear and

$$\|f\|_{(L_p(S;Y))^*} = \|\xi\|_{L_q(S;Y^*)}.$$

Now let us consider the reflexive separable Banach space V with the norm $\|\cdot\|_V$ and the Hilbert space $(H, (\cdot,\cdot)_H)$ with the norm $\|\cdot\|_H$, and for the given spaces let the next conditions be satisfied

$$V \subset H, \ V \text{ is dense in } H,$$

$$\exists \gamma > 0 \colon \|v\|_H \le \|v\|_V \quad \forall v \in V. \tag{7}$$

Due to remark 1 under the given assumptions we may consider the conjugated to H space H^* as a subspace of V^* that is conjugated to V. As V is reflexive then H^* is dense in V^* and

$$\|f\|_{V^*} \le \gamma \|f\|_{H^*} \quad \forall f \in H^*,$$

where $\|\cdot\|_{V^*}$ and $\|\cdot\|_{H^*}$ are the norm in V^* and in H^*, respectively.

Further, we identify the spaces H and H^*. Then we obtain

$$V \subset H \subset V^*$$

with continuous and dense embedding.

Definition 2. The triple of spaces ($V;H;V^*$), that satisfy the latter conditions will be called the evolution triple.

Let us point out that under identification H with H^* and H^* with some subspace of V^*, an element $y \in H$ is identified with some $f_y \in V^*$ and

$$(y, x) = \langle f_y, x \rangle_V \quad \forall x \in V,$$

where $\langle \cdot,\cdot \rangle_V$ is the canonical pairing between V^* and V. Since the element y and f_y are identified then, under condition (7), the pairing $\langle \cdot,\cdot \rangle_V$ will denote the inner product on H $(\cdot,\cdot)$.

We consider p_i, r_i, $i = 1, 2$ such that $1 < p_i \le r_i \le +\infty$, $p_i < +\infty$. Let $q_i \ge r_{i'} \ge 1$ well-defined by

$$p_i^{-1}+q_i^{-1}=r_i^{-1}+r_{i'}^{-1}=1 \quad \forall i=1,2 .$$

Remark that $1/\infty=0$.

Now we consider some Banach spaces that play an important role in the investigation the differential-operator equations and evolution variational inequalities in non-reflexive Banach spaces.

Referring to evolution triples ($V_i;H;V_i^*$) ($i=1,2$) such that

the set $V_1\cap V_2$ is dense in the spaces V_1, V_2 and H (8)

we consider the functional Banach spaces (proposition 2)

$$X_i=X_i(S)=L_{q_i}(S;V_i^*)+L_{r_{i'}}(S;H),\quad i=1,2$$

with norms

$$\|y\|_{X_i}=\inf\left\{\max\left\{\|y_1\|_{L_{q_i}(S;V_i^*)};\ \|y_2\|_{L_{r_{i'}}(S;H)}\right\}\right|$$

$$\left|\ y_1\in L_{q_i}(S;V_i^*),\ y_2\in L_{r_{i'}}(S;H),\ y=y_1+y_2\ \right\},$$

for all $y\in X_i$, and

$$X=X(S)=L_{q_1}(S;V_1^*)+L_{q_2}(S;V_2^*)+L_{r_{2'}}(S;H)+L_{r_{1'}}(S;H)$$

with

$$\|y\|_X=\inf\{\max_{i=1,2}\{\|y_{1i}\|_{L_{q_i}(S;V_i^*)};\ \|y_{2i}\|_{L_{r_{i'}}(S;H)}\}\,|\,y_{1i}\in L_{q_i}(S;V_i^*),$$

$$y_{2i}\in L_{r_{i'}}(S;H),\ i=1,2;\ y=y_{11}+y_{12}+y_{21}+y_{22}\},$$

for each $y\in X$. As $r_i<+\infty$, due to theorem 1 and to theorem 2 the space X_i is reflexive. Analogously, if $\max\{r_1,r_2\}<+\infty$, the space X is reflexive.

Under the latter theorems we identify the conjugated to $X_i(S)$, $X_i^*=X_i^*(S)$, with $L_{r_i}(S;H)\cap L_{p_i}(S;V_i)$, where

$$\|y\|_{X_i^*}=\|y\|_{L_{r_i}(S;H)}+\|y\|_{L_{p_i}(S;V_i)}\quad \forall\, y\in X_i^*,$$

and, respectively, the conjugated to $X(S)$ space $X^*=X^*(S)$ we identify with

$$L_{r_1}(S;H)\cap L_{r_2}(S;H)\cap L_{p_1}(S;V_1)\cap L_{p_2}(S;V_2),$$

where

$$\|y\|_{X^*(S)}=\|y\|_{L_{r_1}(S;H)}+\|y\|_{L_{r_2}(S;H)}+\|y\|_{L_{p_1}(S;V_1)}+\|y\|_{L_{p_2}(S;V_2)}\ \forall\, y\in X^*.$$

On $X(S)\times X^*(S)$ we denote by

$$\langle f,y\rangle=\langle f,y\rangle_S=\int_S (f_{11}(\tau),y(\tau))_H\,d\tau+\int_S (f_{12}(\tau),y(\tau))_H\,d\tau+$$

$$+\int_S \langle f_{21}(\tau), y(\tau)\rangle_{V_1} d\tau + \int_S \langle f_{22}(\tau), y(\tau)\rangle_{V_2} d\tau =$$

$$= \int_S (f(\tau), y(\tau)) d\tau \quad \forall f \in X, \in X^*,$$

where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r_i'}(S;H)$, $f_{2i} \in L_{q_i}(S;V_i^*)$, $i = 1,2$.

Let $V = V_1 \cap V_2$, $\mathcal{F}(V)$ be a filter of all finite-dimensional subspaces from V. As V is separable, there exists countable monotone increasing system of subspaces $\{H_i\}_{i\geq 1} \subset \mathcal{F}(V)$ complete in V, and consequently in H. On H_n we consider inner product induced from H, that we denote again as $(\cdot,\cdot)$. Moreover let $P_n : H \to H_n \subset H$ be orthogonal projection from H on H_n operator:

$$\text{for every } h \in H \quad P_n h = \arg\min_{h_n \in H_n} \| h - h_n \|_H .$$

Definition 3. We say that the triple $(\{H_i\}_{i\geq 1}; V; H)$ satisfies condition (γ), if $\sup_{n\geq 1} \|P_n\|_{L(V,V)} < +\infty$, i.e. there exists such $C \geq 1$ that for every $v \in V$ and $n \geq 1$

$$\|P_n v\|_V \leq C \|v\|_V . \tag{9}$$

Some constructions that satisfy condition (γ) were presented in [7].

Remark 2. It is easy to notice that there exists such complete orthonormal in H vector system $\{h_i\}_{i\geq 1} \subset V$ that for any $n \geq 1$ H_n is a linear capsule stretched on $\{h_i\}_{i=1}^n$. Then condition (γ) means that the given system is a Schauder basis in the space V ([8], p. 403).

Remark 3. Due to the identification of H^* and H it follows that H_n^* and H_n are identified too.

Remark 4. Since $P_n \in \|P_n\|_{L(V,V)}$ for every $n \geq 1$ we get $P_n^* \in \|P_n^*\|_{L(V^*,V^*)}$ and $\|P_n\|_{\mathcal{L}(V,V)} = \|P_n^*\|_{L(V^*,V^*)}$. It is clear that for every $h \in H$ $P_n h = P_n^* h$. Hence, we identify P_n with its conjugate P_n^* for every $n \geq 1$. Then, condition (γ) means that for every $v \in V$ and $n \geq 1$ it results in

$$\|P_n v\|_V \leq C \|v\|_V \quad \text{and} \quad \|P_n v\|_{V^*} \leq C \|v\|_{V^*} . \tag{10}$$

For each $n \geq 1$ we consider the Banach spaces

$$X_n = X_n(S) = L_{q_0}(S; H_n) \subset X, \quad X_n^* = X_n^*(S) = L_{p_0}(S; H_n) \subset X^*,$$

where $p_0 := \max\{r_1, r_2\}$, $q_0^{-1} + p_0^{-1} = 1$ with the natural norms. The space $L_{q_0}(S; H_n)$ is isometrically isomorphic to the conjugate space X_n^* of X_n, moreover, the map

$$X_n \times X_n^* \ni f, x \to \int_S (f(\tau), x(\tau))_{H_n} d\tau = \int_S (f(\tau), x(\tau)) d\tau = \langle f, x\rangle_{X_n}$$

is the duality form on $X_n \times X_n^*$. This statement is correct due to

$$L_{q_0}(S;H_n) \subset L_{q_0}(S;H) \subset L_{r_1'}(S;H) + L_{r_2'}(S;H) + L_{q_1}(S;V_1^*) + L_{q_2}(S;V_2^*).$$

Let us remark that $\langle\cdot,\cdot\rangle_S \mid_{X_n(S)\times X_n^*(S)} = \langle\cdot,\cdot\rangle_{X_n(S)}$.

Proposition 3. For every $n \ge 1$ $X_n = P_n X$, i.e.

$$X_n = \{P_n f(\cdot) \mid f(\cdot) \in X\}.$$

Moreover, if the triple $(\{H_j\}_{j\ge1}; V_i; H)$, $i = 1,2$ satisfies condition (γ) with $C = C_i$, then

for every $f \in X$ and $n \ge 1$ $\|P_n\|_X \le \max\{C_1, C_2\}\|f\|_X$.

Proof. Let us fix an arbitrary number $n \ge 1$. For every $y \in X$ let $y_n(\cdot) := P_n y(\cdot)$, i.e. $y_n(t) = P_n y(t)$ for almost all $t \in S$. Since P_n is linear and continuous on V_1^*, on V_2^* and on H we have that $y_n \in X_n \subset X$. It is immediate that the inverse inclusion is valid.

Now let us prove the second part of this statement. We suppose that condition (γ) holds, let us fix $f \in X$ and $n \ge 1$. Then from condition (γ) it follows that for every $f_{1i} \in L_{r_i'}(S;H)$ and $f_{2i} \in L_{q_i}(S;V_i^*)$ such that $f = f_{11} + f_{12} + f_{21} + f_{22}$ we have

$$\|P_n f_{11}\|_{L_{r_1'}(S;H)} + \|P_n f_{12}\|_{L_{r_2'}(S;H)} + \|P_n f_{21}\|_{L_{q_1}(S;V_1^*)} +$$

$$+\|P_n f_{22}\|_{L_{q_2}(S;V_2^*)} = \left(\int_S \|P_n f_{11}(\tau)\|_H^{r_1'} d\tau\right)^{\frac{1}{r_1'}} + \left(\int_S \|P_n f_{12}(\tau)\|_H^{r_2'} d\tau\right)^{\frac{1}{r_2'}} +$$

$$+\left(\int_S \|P_n f_{21}(\tau)\|_{V_1^*}^{q_1} d\tau\right)^{\frac{1}{q_1}} + \left(\int_S \|P_n f_{22}(\tau)\|_{V_2^*}^{q_2} d\tau\right)^{\frac{1}{q_2}} \le$$

$$\le \left(\int_S \|f_{11}(\tau)\|_H^{r_1'} d\tau\right)^{\frac{1}{r_1'}} + \left(\int_S \|f_{12}(\tau)\|_H^{r_2'} d\tau\right)^{\frac{1}{r_2'}} +$$

$$+ C_1\left(\int_S \|f_{21}(\tau)\|_{V_1^*}^{q_1} d\tau\right)^{\frac{1}{q_1}} + C_2\left(\int_S \|f_{22}(\tau)\|_{V_2^*}^{q_2} d\tau\right)^{\frac{1}{q_2}} \le$$

$$\le \max\{C_1, C_2\}\Big(\|f_{11}\|_{L_{r_1'}(S;H)} + \|f_{12}\|_{L_{r_2'}(S;H)} +$$

$$+ \|f_{21}\|_{L_{q_1}(S;V_1^*)} + \|f_{22}\|_{L_{q_2}(S;V_2^*)} \Big),$$

because $C_1, C_2 \geq 1$. Therefore, due to the definition of norm in X we complete the proof.

Proposition 4. For every $n \geq 1$ it results in $X_n^* = P_n X^*$, i.e.

$$X_n^* = \{P_n y(\cdot) \mid y(\cdot) \in X^*\},$$

and

$$\langle f, P_n y\rangle = \langle f, y\rangle \quad \forall\, y \in X^* \text{ and } f \in X_n.$$

Furthermore, if the triple $\left(\{H_j\}_{j\geq 1}; V_i; H\right)$, $i = 1,2$ satisfies condition (γ) with $C = C_i$, then we get

$$\|P_n\|_{X^*} \leq \max\{C_1, C_2\} \|y\|_{X^*} \quad \forall\, y \in X^* \text{ and } n \geq 1.$$

Proof. For every $y \in X^*$ we set $y_n(\cdot) := P_n y(\cdot)$, i.e. $y_n(t) = P_n y(t)$ for a.e. $t \in S$. As the operator P_n is linear and continuous on V_1, on V_2 and on H we have that $y_n \in X^*{}_n \subset X^*$. The inverse inclusion is obvious.

Due to condition (γ) and to the definition of $\|\cdot\|_{L_{p_i}(S;V_i)}$ and $\|\cdot\|_{L_{r_i}(S;H)}$ it follows that

$$\|y_n\|_{L_{p_i}(S;V_i)} \leq C_i \cdot \|y\|_{L_{p_i}(S;V_i)} \quad \text{and} \quad \|y_n\|_{L_{r_i}(S;H)} \leq \|y\|_{L_{r_i}(S;H)}.$$

Thus $\|y_n\|_{X^*} \leq \max\{C_1, C_2\} \|y\|_{X^*}$.

Now let us show that for every $f \in X_n$

$$\langle f, y_n\rangle = \langle f, y\rangle.$$

As $f \in L_{p_0}(S; H_n)$, then we have

$$\langle f, y\rangle = \int_S (f(\tau), y(\tau)) d\tau = \int_S (f(\tau), P_n y(\tau)) d\tau =$$

$$= \int_S (f(\tau), y_n(\tau)) d\tau = \langle f, y_n\rangle,$$

because for every $n \geq 1$, $h \in H$ and $v \in H_n$ it results in

$$(h - P_n h, v) = (h - P_n h, v)_H = 0.$$

The proposition is proved.

Proposition 5. Under the condition $\max\{r_1, r_2\} < +\infty$ the set $\bigcup_{n\geq 1} X_n^*$ is dense in $(X^*, \|\cdot\|_{X^*})$.

Proof. a) At first we prove that the set $L_\infty(S;V)$ is dense in space

$$(X^*, \|\cdot\|_{X^*}).$$

Let us fix $x \in X^*$.

Then for every $n \ge 1$ we consider

$$x_n(t) := \begin{cases} x(t) & \|x(t)\|_V \le n, \\ 0, & \text{elsewhere.} \end{cases} \quad (11)$$

Obviously $x_n \in L_\infty(S;V)$. The continuous embedding of V into H assures the existence of some positive γ such that for $i = 1,2$ and a.e. $t \in S$ we have

$$\left.\begin{aligned} \|x_n(t) - x(t)\|_H \le \gamma \|x_n(t) - x(t)\|_V \to 0, \\ \|x_n(t) - x(t)\|_{V_i} \le \|x_n(t) - x(t)\|_V \to 0, \text{ as } n \to \infty, \end{aligned}\right\} \quad (12)$$

$$\|x_n(t)\|_H \le \|x(t)\|_H, \ \|x_n(t)\|_{V_i} \le \|x(t)\|_{V_i}. \quad (13)$$

Further let us set

$$\phi_H^n(t) = \|x_n(t) - x(t)\|_H^{p_0}, \quad \phi_{V_i}^n(t) = \|x_n(t) - x(t)\|_{V_i}^{p_i}.$$

So, from (12) and (13) we obtain

$$\phi_H^n(t) \to 0, \quad \phi_{V_i}^n(t) \to 0 \ \text{ as } n \to \infty \quad \text{for a.e. } t \in S \quad (14)$$

and for almost every $t \in S$

$$|\phi_H^n(t)| \le 2^{p_0} \|x(t)\|_H^{p_0} =: \phi_H(t), \quad |\phi_{V_i}^n(t)| \le 2^{p_i} \|x(t)\|_{V_i}^{p_i} =: \phi_{V_i}(t). \quad (15)$$

Since $x \in X^*$, then $\phi_H, \phi_{V_1}, \phi_{V_2} \in L_1(S)$. Thus, due to (14) and (15), we can apply the Lebesgue theorem with integrable majorants ϕ_H, ϕ_{V_1} and ϕ_{V_2} respectively. Hence it follows that $\phi_H^n \to \bar{0}$ and $\phi_{V_i}^n \to \bar{0}$ in $L_1(S)$ as $i = 1,2$. Consequently $\|x_n - x\|_{X^*} \to 0$ as $n \to \infty$.

b) Further, for some linear variety L from V we set

$$\Upsilon(L) := \{y \in (S \to L) \mid y \text{ is a simple function}\}$$

([6], p.152). Let us show that set $\Upsilon(V)$ is dense in the normalized space $(L_\infty(S,V), \|\cdot\|_{X^*})$. Let be x fixed in $L_\infty(S,V)$; so, there exists a sequence $\{x_n\}_{n\ge1} \subset \Upsilon(V)$ such that

$$x_n(t) \to x(t) \text{ in } V \text{ as } n \to \infty \text{ for a.e. } t \in S. \quad (16)$$

Since $x \in L_\infty(S,V)$ we have $\operatorname*{ess\,sup}_{t\in S} \|x(t)\|_V =: c < +\infty$. For every $n \ge 1$ let us introduce

$$y_n(t) := \begin{cases} x_n(t), & \|x_n(t)\|_V \le 2c, \\ \bar{0}, & \text{else.} \end{cases} \quad (17)$$

From (16) and (17) it follows that $y_n \in \Upsilon(V)$ as $n \ge 1$ and moreover,

$$y_n(t) \to x(t) \text{ in } V \text{ as } n \to \infty \text{ for a.e. } t \in S.$$

Hence, taking into account $V \subset H$, as $i=1,2$ and for a.e. $t \in S$ we obtain the following convergences

$$y_n(t) \to x(t) \text{ in } H, \quad y_n(t) \to x(t) \text{ in } V_1, \quad y_n(t) \to x(t) \text{ in } V_2 \quad \text{as } n \to \infty .$$

As in a), assuming

$$\phi_H \equiv \phi_{V_1} \equiv \phi_{V_2} \equiv \max\{(3c)^{p_1}, (3c)^{p_2}, (3c\gamma)^{p_0}\} \in L_1(S))$$

we obtain that $y_n \to x$ in X^* as $n \to \infty$. So, $\Upsilon(V)$ is dense in

$$(L_\infty(S,V), \|\cdot\|_{X^*}).$$

c) Since the set $\operatorname{span}\{h_n\}_{n\ge1} = \bigcup_{n\ge1} H_n$ is dense in $(V, \|\cdot\|_V)$ and $V \subset H$ with continuous embedding it follows that the set

$$\Upsilon\left(\bigcup_{n\ge1} H_n\right) = \bigcup_{n\ge1} \Upsilon(H_n) \text{ is dense in } (\Upsilon(V), \|\cdot\|_{X^*}).$$

In order to complete the proof we point out that for every $n \ge 1$ $\Upsilon(H_n) \subset X^*{}_n$. The proposition is proved.

Now we consider Banach space $W^* = \{y \in X^* \mid y' \in X\}$ with the norm

$$\|y\|_{W^*} = \|y\|_{X^*} + \|y'\|_X ,$$

where the derivative y' of an element $y \in X^*$ is in the sense of the scalar distribution space $\mathcal{D}^*(S;V^*) = \mathcal{L}(\mathcal{D}(S);V_w^*)$, where V_w^* be equals to V^* with topology $\sigma(V^*;V)$ [9].

Together with $W^* = W^*(S)$ we consider the Banach space

$$W_i^* = W_i^*(S) = \{y \in L_{p_i}(S;V_i) \,|\, y' \in X(S)\}, \quad i=1,2,$$

with the norm

$$\|y\|_{W_i^*} = \|y\|_{L_{p_i}(S;V_i)} + \|y'\|_X \quad \forall\, y \in W_i^* .$$

We also consider the space $W_0^* = W_0^*(S) = W_1^*(S) \cap W_2^*(S)$ with the norm

$$\|y\|_{W_0^*} = \|y\|_{L_{p_1}(S;V_1)} + \|y\|_{L_{p_2}(S;V_2)} + \|y'\|_X \quad \forall\, y \in W_0^* .$$

The space W^* is continuously embedded in W_i^* for $i = \overline{0,2}$.

Theorem 3. It results in $W_i^* \subset C(S;V^*)$ with continuous embedding for $i = \overline{0,2}$.

Proof. Let $i=1,2$ be fixed, $y \in W_i^*$ and $\forall t_0, t \in S$ we set $\xi(t) = \int_{t_0}^{t} y'(\tau)d\tau$ which has sense in the virtue of the local integrability of y'. It is obvious that

$$\|\xi(t)-\xi(s)\|_{V^*} \le \int_t^s \|y'(\tau)\|_{V^*} d\tau \quad \forall\, s \ge t$$

from which follows $\xi \in C(S;V^*)$. Then $\xi' = y'$, it means that $y(t) = \xi(t) + z$ for a.e. $t \in S$ and some $z \in V^*$. Therefore, the function y also belongs to $C(S;V^*)$. Note, that S is compact. Then in virtue of $X \subset L_1(S;V^*)$ we obtain

$$\|\xi(t)\|_{V^*} \le \int_S \|y'(\tau)\|_{V^*} d\tau \le k\|y'\|_X \quad \forall\, t \in S .$$

Then due to the continuity of embedding $V_i \subset V^*$ we have

$$\|z\|_{V^*} (\mathrm{mes}\,(S))^{1/p_i} = (\int_S \|z\|_{V^*}^{p_i} ds)^{1/p_i} = \|y - \xi\|_{L_{p_i}(S;V^*)} \le$$

$$\le k_1(\|y\|_{L_{p_i}(S;V^*)} + \|\xi\|_{C(S;V^*)}) \le k_2(\|y\|_{L_{p_i}(S;V_i)} + \|y'\|_X) \qquad (18)$$

where k_2 does not depend on $y \in W_i^*$.

Now let $y \in W_0^* \subset C(S;V^*)$. In virtue of (18) for $i = 1,2$ there exists $k_3 \ge 0$ that $\|y\|_{C(S;V^*)} \le k_3 \|y\|_{W_0}$ for all $y \in W_0^*$.

Remark 5. From the definition of norms in the spaces W^* and W_0^* we obtain $W^* \subset C(S;V^*)$ with continuous embedding for the compact S in the natural topology of the space W^*.

Theorem 4. The set $C^1(S;V) \cap W_0^*$ is dense in W_0^*.

Proof. We prove this statement for more general case. At the beginning we suppose $S = \mathbb{R}$. Let us choose such a function $K \in C_0^\infty(S)$ that $\int_S K(\tau)d\tau = 1$ and use the Sobolev mid-value method. Let us set for definiteness

$$K(\tau) = \begin{cases} \mu \exp\{-\dfrac{\tau^2}{\tau^2 - 1}\} & \text{for} \quad |\tau| \le 1, \\ 0 & \text{for} \quad |\tau| > 1, \end{cases}$$

where μ is the constant of normalization and suppose $K_n(\tau) = nK(n\tau)$ for every $\tau \in S$ and $n \ge 1$. It is obvious that $K_n \in C_0^\infty(S)$ and

$$\int_S K_n(\tau)d\tau = 1 \quad \forall\, n \ge 1 .$$

For every $y \in W_0^*$ let us define the sequence of functions

$$y_n(t) = \int_S K_n(t-\tau)y(\tau)d\tau. \qquad (19)$$

It is easy to check that $y_n \in C^1(S;V)$ and

$$y_n' = \int_S K_n'(t-\tau)y(\tau)d\tau = \int_S K_n(t-\tau)y'(\tau)d\tau. \tag{20}$$

Besides $y_n \in L_{p_i}(S;V_i)$ and $y_n \to y$ in $L_{p_i}(S;V_i)$ for ($i=1,2$). The last follows from the inequality $\|y_n\|_{L_{p_i}(S;V_i)} \le \|K\|_{L_1(S)}\|y\|_{L_{p_i}(S;V_i)}$ and from following estimations:

$$\|y_n - y\|^{p_i}_{L_{p_i}(S;V_i)} = \int_S \| \int_{t-1/n}^{t+1/n} K_n(t-\tau)(y(\tau)-y(t))d\tau\|^{p_i}_{V_i} dt \le$$

$$\le \int_S \{ (\int_{-1/n}^{1/n} |K_n(s)|^{q_i} ds)^{p_i/q_i} \int_{-1/n}^{1/n} \|y(t+s)-y(t)\|^{p_i}_{V_i} ds\} dt \le$$

$$\le \frac{n}{2}(2\mu)^{p_i} \int_{-1/n}^{1/n} (\int_S \|y(t+s)-y(t)\|^{p_i}_{V_i} dt) ds .$$

Pointing out that for arbitrary $y \in L_{p_i}(S;V_i)\,(1 \le p_i < \infty)$ and for every h the function

$$y_h(t) = \begin{cases} y(t+h) & \text{for} \quad t+h \in S, \\ 0 & \text{for} \quad t+h \notin S \end{cases}$$

belongs to $L_{p_i}(S;V_i)$ and $\|y_h - y\|_{L_{p_i}(S;V_i)} \to 0$ as $h \to 0$ [6, lemma IV.1.5], then

$$\lim_{n\to\infty} \|y_n - y\|^{p_i}_{L_{p_i}(S;V_i)} \le \lim_{n\to\infty} (2\mu)^{p_i} \sup_{|s|\le 1/n} \|y_s - y\|^{p_i}_{L_{p_i}(S;V_i)} = 0 \quad \text{for} \quad i = 1,2 .$$

Now we prove the convergence of derivatives. Let $y' \in X$ and $y' = \xi_1 + \xi_2 + \eta_1 + \eta_2$ where $\xi_i \in L_{q_i}(S;V_i^*)$, $\eta_i \in L_{r_i'}(S;H)$, $i=1,2$. By the analogy with (19) we suppose

$$\xi_{n,i}(t) = \int_S K_n(t-\tau)\xi_i(\tau)d\tau, \quad \eta_{n,i}(t) = \int_S K_n(t-\tau)\eta_i(\tau)d\tau \quad \text{for} \quad i=1,2 .$$

Then in virtue of (20) by the analogy to the previous case, $y_n' = \xi_{n,1} + \xi_{n,2} + \eta_{n,1} + \eta_{n,2}$ and besides $\xi_{n,i} \to \xi_i$ in $L_{q_i}(S;V_i^*)$ and $\eta_{n,i} \to \eta_i$ in $L_{r_i'}(S;H)$ for $i=1,2$. By definition of $\|\cdot\|_X$, it follows

$$\lim_{n\to\infty} \|y_n' - y'\|_X \le \lim_{n\to\infty} \max \left\{ \|\xi_{n,1} - \xi\|_{L_{q_1}(S;V_1^*)}; \|\xi_{n,2} - \xi\|_{L_{q_2}(S;V_2^*)}; \right.$$

$$\left. \|\eta_{n,1} - \eta\|_{L_{r_1'}(S;H)}; \|\eta_{n,2} - \eta\|_{L_{r_2'}(S;H)} \right\} = 0$$

From here we conclude that for every $n \ge 1$ $y_n \in C^1(S;V) \cap W_0^*$ and the sequence $\{y_n\}_{n\ge 1}$ converges to $y \in W_0^*$ in W_0^*.

Now let us consider the case when S is semi-bounded. Without loss of generality we suppose $S=[0,\infty)$. For $y \in W_0^* = W_0^*(S)$ we put $y_h(t) = y(t+h)$ for every $h>0$. Then, in virtue of [6, lemma IV.1.5] it is easy to show that for $i=1,2$ $y_h \to y$ in $L_{p_i}(S;V_i)$ and $y_h' \to y'$ in X as $h \to 0+$. Remark that $y_h \in W_0^*$. To complete the proof it is sufficient to show that for every fixed $y \in W_0^*(S)$ and for $h>0$ the element $y_h \in W_0^*$ can be sufficiently exactly approximated by the functions from $C^1(S;V) \cap W_0^*$.

For some $y \in W_0^*(S)$ and $h>0$ let us consider the function

$$\xi(t) = \begin{cases} \varphi(t)\, y(t+h) & \text{for} \quad t \ge -h, \\ 0 & \text{for} \quad t < -h, \end{cases}$$

where $\varphi \in C^1(\mathbb{R})$, $\varphi(t)=1$ if $t \ge -\dfrac{h}{2}$ and $\varphi(t)=0$ if $t<-h$. Then for every $t \ge 0$ $\xi(t) = y_h(t)$ and due to definition of derivative in sense of scalar distribution space $D^*(S;V^*)$ it follows that

$$\xi'(t) = \begin{cases} \varphi'(t) y(t+h) + \varphi(t) y'(t+h) & \text{for} \quad t \ge -h, \\ 0 & \text{for} \quad t < -h. \end{cases}$$

Let us prove that $\xi \in W_0^*(\mathbb{R})$. Since $y_h \in W_0^*(S)$ we have $\xi|_{[0,\infty)} \in X^*(S)$. Because of $\xi|_{(-\infty;-h)} = 0$ it remains to consider the section $[-h,0)$.

From $\sup\limits_{s\in[-h,0)} |\varphi(s)| = 1$ we have

$$\int_{-h}^{0} \|\xi(s)\|_{V_i}^{p_i} ds \le \int_{-h}^{0} |\varphi(s)|^{p_i} \, \|y(s+h)\|_{V_i}^{p_i} ds \le$$

$$\le \int_{-h}^{0} \|y(s+h)\|_{V_i}^{p_i} ds = \int_{0}^{h} \|y(\tau)\|_{V_i}^{p_i} d\tau \quad (i=1,2).$$

Thus, $\xi|_{[-h,0)} \in L_{p_i}([-h,0);V_i)$ for $i=1,2$. Similarly we can prove that $\xi' \in X(\mathbb{R})$. So, $\xi \in W_0^*(\mathbb{R})$ and in virtue of the previous case there exists a sequence of elements $\xi_n \in C^1(\mathbb{R};V) \cap W_0^*(\mathbb{R})$ that converges to ξ in $W_0^*(\mathbb{R})$.

Now we set $\zeta_n = \xi_n|_S \in C^1(S;V) \cap W_0^*(S)$ for every $n \ge 1$. Here $\zeta_n \to y_h$ in $W_0^*(S)$ as $n \to \infty$, because $\xi|_S = y_h$.

Let us consider, at last, the case when S is bounded. For every $y \in W_0^*(S)$ (where $S=[a,b]$, $a<b$) we put

$$\xi(t) = \begin{cases} \varphi(t) y(t) & \text{for} \quad t \in [a,b], \\ 0 & \text{for} \quad t > b, \end{cases}$$

$$\eta(t)=\begin{cases}(1-\varphi(t))y(t) & \text{for} \quad t\in[a,b],\\ 0 & \text{for} \quad t<a.\end{cases}$$

Let φ be such function from $C^1(S)$ that $\varphi(t)=0$ in some neighborhood of the point b and $\varphi(t)=1$ in some neighborhood of the point a. Note that $y(t)=\xi(t)+\eta(t)$ for all $t\in S$. It is easy to check that $\xi\in W_0^*([a,\infty))$ and $\eta\in W_0^*((-\infty,b])$. Therefore, due to the previous case, there exist such sequences

$$\{\xi_n\}_{n\ge 1}\subset C^1([a,\infty);V)\cap W_0^*([a,\infty))$$

and

$$\{\eta_n\}_{n\ge 1}\subset C^1((-\infty,b);V)\cap W_0^*((-\infty,b)),$$

that

$$\xi_n\to\xi \text{ in } W_0^*([a,\infty)) \text{ and } \eta_n\to\eta \text{ in } W_0^*((-\infty,b)) \text{ as } n\to\infty .$$

So, $(\xi_n+\eta_n)|_S\to y$ in $W_0^*(S)$.

The theorem is proved.

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