НОВІ МЕТОДИ В СИСТЕМНОМУ АНАЛІЗІ, ІНФОРМАТИЦІ ТА ТЕОРІЇ ПРИЙНЯТТЯ РІШЕНЬ

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ON SOME APPROXIMATIONS AND MAIN TOPOLOGICAL DESCRIPTIONS FOR SPECIAL CLASSES OF FRECHET SPACES WITH INTEGRABLE DERIVATIVES

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We consider some classes of Frechet spaces with integrable derivatives. Important compactness lemmas for nonreflexive spaces are obtained. Some main topological properties for the given spaces are obtained.

Method of monotony and method of compactness represent fundamental approaches to study nonlinear differential-operator equations, evolutionary inclusions and variational inequalities in Banach spaces. The general idea is the following: using the corresponding approximation scheme, the approximate solutions of a problem are constructed, for them the approaching a priori estimates are established, at last they prove the existence of sequence of approximate solutions, that converges to the exact solution of problem. In many cases the aim is obtained by using both a method of compactness and a method of monotonicity.

Now we introduce some constructions to prove the convergence of Faedo–Galerkin method for a global solvability of differential-operational equations, inclusions and evolution variation inequalities with w_λ–pseudomonotone maps [1, 2, 3, 4]. Moreover, we obtain a new theorems of compact embedding for Frechet spaces, suggested by researches of differential-operational inclusions in function spaces.

For a pair of Banach spaces X, Y the notation $X \subset Y$ further will mean the embedding both in the set-theory sense and in the topological sense.

Let Y be some Banach space; Y^* be its topologically conjugated space; I be some compact time interval. We consider the classes of functions defined on I and imagines in Y (or in Y^*).

The set $L_p(I;Y)$ of measured by Bochner functions (see [5]) as $1 \le p \le +\infty$ with the natural linear operations is a Banach space with the norm

$$\|y\|_{L_p(I;Y)} = \left(\int_I \|y(t)\|_Y^p dt \right)^{1/p}.$$

As $p = +\infty$ we have a Banach space $L_\infty(I;Y)$ with the norm

$$\|y\|_{L_\infty(I;Y)} = \operatorname*{ess\,max}_{t\in I} \|y(t)\|_Y .$$

The next theorem proves that under the reflexivity or separability of Y the conjugated to $L_p(I;Y)$, $1 \le p < +\infty$, $(L_p(I;Y))^*$ may be identify with $L_q(I;Y^*)$, $p^{-1} + q^{-1} = 1$.

Theorem 1. (Rietz) If the space Y is reflexive or separable and $1 \le p < +\infty$, then each element $f \in (L_p(I;Y))^*$ gives the unique representation

$$f(y) = \int_I \langle \xi(t), y(t)\rangle_Y \, dt \quad \text{for every} \quad y \in L_p(I;Y)$$

with the function $\xi \in L_q(I;Y^*)$, $p^{-1} + q^{-1} = 1$. The correspondence $f \to \xi$, $f \in (L_p(I;Y))^*$ is linear and

$$\|f\|_{(L_p(I;Y))^*} = \|\xi\|_{L_q(I;Y^*)} .$$

Now let us consider the reflexive separable Banach space V with the norm $\|\cdot\|_V$ and the Hilbert space $(H, (\cdot,\cdot)_H)$ with the norm $\|\cdot\|_H$, and let the next conditions are valid:

$$\begin{aligned} &V \subset H, \quad V \text{ is dense in } H, \\ &\exists \gamma > 0 : \|v\|_H \le \gamma \|v\|_V \quad \forall v \in V. \end{aligned} \tag{1}$$

Under these assumptions we may consider the conjugated to H, H^*, as a subspace V^* that is conjugated to V. As V is reflexive then H^* is dense in V^* and

$$\|f\|_{V^*} \le \gamma \|f\|_{H^*} \quad \forall f \in H^*,$$

where $\|\cdot\|_{V^*}$ is the norm in V^*, $\|\cdot\|_{H^*}$ is the norm in H^*.

Further, we identify the spaces H and H^*. Then we obtain $V \subset H \subset V^*$ with continuous and dense embedding.

Definition 1. The triple of spaces $(V; H; V^*)$, that satisfies the latter conditions is called the evolution triple.

Let us note that under identification of H with H^* and H^* with some subspace of V^*, an element $y \in H$ coincides with some $f_y \in V^*$ and $(y, x) = \langle f_y, x\rangle_V$ $\forall x \in V$, where $\langle \cdot,\cdot \rangle_V$ is the canonical pairing between V^* and V. Since the element y and f_y are identified then, under condition (1), the pairing $\langle \cdot,\cdot \rangle_V$ and the inner product on H will be denoted the same notation $(\cdot,\cdot)$.

By the analogy with [7] we consider p_i, r_i, $i=1,2$ such that $1<p_i\le$ $\le r_i\le+\infty$, $p_i<+\infty$. Let $q_i\ge r_i\ge 1$ well-defined defined by $p_i^{-1}+q_i^{-1}=$ $=r_i^{-1}+r_{i'}^{-1}=1$ $\forall i=1,2$. Remark that $1/\infty=0$.

Now we consider some Banach spaces that play an important role in the investigation on differential-operator equations and evolution variation inequalities in non-reflexive Banach spaces.

For evolution triples $(V_i;H;V_i^*)$ ($i=1,2$) such that

$$\text{the set } V_1\cap V_2 \text{ is dense in the spaces } V_{1,2} \text{ and } H \tag{2}$$

and for some compact time interval we consider the functional Banach spaces

$$X_i(I)=L_{q_i}(I;V_i^*)+L_{r_{i'}}(I;H),\quad i=1,2$$

with norms

$$\|y\|_{X_i(I)}=\inf\{\max\{\|y_1\|_{L_{q_i}(I;V_i^*)};\|y_2\|_{L_{r_{i'}}(I;H)}\}\mid y_1\in$$

$$\in L_{q_i}(I;V_i^*),y_2\in L_{r_{i'}}(I;H),y=y_1+y_2\},$$

for all $y\in X_i(I)$, and $X(I)=L_{q_1}(I;V_1^*)+L_{q_2}(I;V_2^*)+L_{r_{2'}}(I;H)+L_{r_{1'}}(I;H)$ with

$$\|y\|_{X(I)}=\inf\{\max_{i=1,2}\{y_{1i}\|_{L_{q_i}(I;V_i^*)};\|y_{2i}\|_{L_{r_{i'}}(I;H)}\}\,|y_{1i}\in L_{q_i}(I;V_i^*),$$

$$y_{2i}\in L_{r_{i'}}(I;H),\, i=1,2;\; y=y_{11}+y_{12}+y_{21}+y_{22}\},$$

for each $y\in X$. We remark that if $r_i<+\infty$ then the space $X_i(I)$ is reflexive. Analogously, if $\max\{r_1,r_2\}<+\infty$, then the space $X(I)$ is reflexive.

Following by [7] we identify $X_i^*(I)$, conjugated to $X_i(I)$, with $L_{r_i}(I;H)\cap L_{p_i}(I;V_i)$, where $\|y\|_{X_i^*(I)}=\|y\|_{L_{r_i}(I;H)}+\|y\|_{L_{p_i}(I;V_i)}$ $\forall y\in$ $\in X_i^*(I)$ and $X^*(I)$, conjugated to $X(I)$, with $L_{r_1}(I;H)\cap L_{r_2}(I;H)\cap$ $\cap L_{p_1}(I;V_1)\cap L_{p_2}(I;V_2)$, where $\|y\|_{X^*(I)}=\|y\|_{L_{r_1}(I;H)}+\|y\|_{L_{r_2}(I;H)}+$ $+\|y\|_{L_{p_1}(I;V_1)}+\|y\|_{L_{p_2}(I;V_2)}$ $\forall y\in X^*(I)$.

On $X(I)\times X^*(I)$ we denote the duality form by the rule:

$$\langle f,y\rangle_I=\int_I(f_{11}(\tau),y(\tau))_H\,d\tau+\int_I(f_{12}(\tau),y(\tau))_H\,d\tau+$$

$$+\int_I\langle f_{21}(\tau),y(\tau)\rangle_{V_1}\,d\tau+\int_I\langle f_{22}(\tau),y(\tau)\rangle_{V_2}\,d\tau=$$

$$= \int_I (f(\tau), y(\tau)) d\tau \quad \forall f \in X, \in X^*,$$

where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r_i'}(I; H)$, $f_{2i} \in L_{q_i}(I; V_i^*)$, $i = 1,2$.

Let $V = V_1 \cap V_2$, $\mathcal{F}(V)$ be a filter of all finite-dimensional subspaces from V. As V is separable, there exists a countable monotone increasing system of subspaces $\{H_i\}_{i \ge 1} \subset \mathcal{F}(V)$ complete in V, and consequently in H. On H_n we consider inner product induced from H, that we denote again as $(\cdot,\cdot)$. Moreover let $P_n : H \to H_n \subset H$ be the operator of orthogonal projection from H on H_n:

$$\text{for every} \quad h \in H \quad P_n h = \arg\min_{h_n \in H_n} \|h - h_n\|_H .$$

Definition 2. We say that the triple $(\{H_i\}_{i \ge 1}; V; H)$ satisfies condition (γ), if $\sup\limits_{n \ge 1} \|P_n\|_{\mathcal{L}(V,V)} < +\infty$, i.e. there exists such $C \ge 1$ that for every $v \in V$ and $n \ge 1$

$$\|P_n v\|_V \le C \|v\|_V . \tag{3}$$

Some constructions that satisfy the above condition were presented in [6].

Remark 1. It is easy to check that there exists such complete orthonormal in H vector system $\{h_i\}_{i \ge 1} \subset V$ such that for every $n \ge 1$ H_n is a linear capsule stretched on $\{h_i\}_{i=1}^n$. Then condition (γ) means that the system is a Schauder basis in the space V (see [9], p.403).

Remark 2. From the identification between H^* and H it follows that H_n^* and H_n are also identified.

Remark 3. In virtue of $P_n \in \mathcal{L}(V,V)$ for every $n \ge 1$ the conjugate operator $P_n^* \in \mathcal{L}(V^*, V^*)$ and $\|P_n\|_{\mathcal{L}(V,V)} = \|P_n^*\|_{\mathcal{L}(V^*,V^*)}$. It is obvious that for every $h \in H$ $P_n h = P_n^* h$. Hence, we identify P_n with its conjugate P_n^* for every $n \ge 1$. Then, the condition (γ) will mean that for every $v \in V$ and $n \ge 1$

$$\|P_n v\|_V \le C \cdot \|v\|_V \text{ and } \|P_n v\|_{V^*} \le C \|v\|_{V^*} . \tag{4}$$

Let us denote by S a subset of a real line which can be presented as no more than numerable join of convex sets in $\mathbb{R}$. We denote by $BC(S) = \{I_\alpha\}_{\alpha \in \Theta}$ the family of all convex bounded sets from S, distinct from a point.

Remark 4. Notice that $\Theta = \Theta(S)$, i.e. the set of indexes depends on the set S. Further, Θ will mean $\Theta(S)$.

Furthermore we set

$$X^{\text{loc}} = \{y : S \to V^* \mid \forall \alpha \in \Theta \quad y|_{I_\alpha} \in X(I_\alpha)\},$$

where $V^* = V_1^* + V_2^*$. In this space the local base of topology is the following:

$$\mathcal{B}_X := \left\{ \bigcap_{k=1}^{n} V(\alpha_k, \varepsilon_k) \,\middle|\, \varepsilon_k > 0, \ \alpha_k \in \Theta, \ k = \overline{1,n}, \quad n \ge 1 \right\},$$

where for every $\alpha \in \Theta$ and $\varepsilon > 0$:

$$V(\alpha, \varepsilon) = \left\{ u \in X^*_{\text{loc}} \mid \|u\|_{X^*(I_\alpha)} < \varepsilon \right\}.$$

Lemma 1 $\mathcal{B}_X$ is local base of some topology τ_X in X^{loc}, which converts the given space in a separable locally convex linear topological space and, moreover,

a) τ_X is compatible with the set of seminorms

$$\{\rho_\alpha(\cdot) = \|\cdot\|_{X(I_\alpha)}\}_{\alpha\in\Theta} \ \text{ on } \ X^{\text{loc}}; \tag{5}$$

b) a set $E \subset X^{\text{loc}}$ is bounded only when $\forall \alpha \in \Theta$ ρ_α is bounded on E.

Proof. We prove the system of seminorms $\{\rho_\alpha\}_{\alpha\in\Theta}$ divides points on X^{loc}. Let $u \in X^{\text{loc}} \setminus \{\bar{0}\}$, then $\lambda(t \in S \mid u(t) \ne \bar{0}) > 0$, where λ is Lebesgue measure on $\mathbb{R}$. Because of S is a subset of a real line which can be presented as no more than numerable join of convex sets in $\mathbb{R}$, we have $\exists \alpha_0 \in \Theta$: $\|u\|_{X(I_{\alpha_0})} > 0$. From here it follows $\rho_{\alpha_0}(u) > 0$, as it was to be shown.

From [12, theorem 1.37] it follows, that the system of seminorms $\{\rho_\alpha\}_{\alpha\in\Theta}$ generates some locally convex topology τ_X on X^{loc}, which converts the given space in locally convex linear topological space, whose local base we obtain by final intersections of such sets:

$$\left\{ V(I_\alpha, \varepsilon) = \{u \in X^{\text{loc}} \mid \rho_\alpha(u) < \varepsilon\} \mid \alpha \in \Theta, \ \varepsilon > 0 \right\}.$$

The statement b) follows from the same theorem.

The lemma is proved.

Let

$$X^*_{\text{loc}} = \left\{ y : S \to V^* \mid \forall \alpha \in \Theta \quad y|_{I_\alpha} \in X^*(I_\alpha) \right\}.$$

In this space the local base of topology is the following:

$$\mathcal{B}_{X^*} := \left\{ \bigcap_{k=1}^{n} V(\alpha_k, \varepsilon_k) \,\middle|\, \varepsilon_k > 0, \ \alpha_k \in \Theta, \ k = \overline{1,n}, \quad n \ge 1 \right\}$$

where for every $\alpha \in \Theta$ and $\varepsilon > 0$:

$$V(\alpha, \varepsilon) = \left\{ u \in X^*_{\text{loc}} \mid \|u\|_{X^*(I_\alpha)} < \varepsilon \right\}.$$

Lemma 2. $\mathcal{B}_{X^*}$ is local base of some topology τ_{X^*} in X^*_{loc}, which converts the given space in a separable locally convex linear topological space and, moreover,

a) τ_{X^*} is compatible with the set of seminorms

$$\{\rho_\alpha(\cdot) = \|\cdot\|_{X^*(I_\alpha)}\}_{\alpha\in\Theta} \text{ on } X^*_{\text{loc}}; \tag{6}$$

b) a set $E \subset X^*_{\text{loc}}$ is bounded only when $\forall\,\alpha\in\Theta$ ρ_α is bounded on E.

Proof. As well as in lemma 1 it is enough to show that system of seminorms $\{\rho_\alpha\}_{\alpha\in\Theta}$ divides points on X^*_{loc}. Let $u \in X^*_{\text{loc}} \setminus \{\bar{0}\}$, then $\lambda(t\in S \mid u(t) \neq \neq \bar{0}) > 0$. Because of S is a subset of a real line which can be presented as no more than numerable join of convex sets in $\mathbb{R}$, we have $\exists\alpha_0\in\Theta$: $\|u\|_{X^*(I_{\alpha_0})} > 0$. From here it follows $\rho_{\alpha_0}(u) > 0$, as it was to be shown.

From [12, theorem 1.37] it follows, that the given system of seminorms $\{\rho_\alpha\}_{\alpha\in\Theta}$ generates some locally convex topology τ_{X^*} on X^*_{loc}, which converts the given space in locally convex linear topological space, whose local base we obtain by find intersections of such sets:

$$\left\{V(I_\alpha,\varepsilon) = \{u\in X^*_{\text{loc}} \mid \rho_\alpha(u) < \varepsilon\} \mid \alpha\in\Theta,\ \varepsilon > 0\right\}.$$

The statement b) follows from same theorem.

The lemma is proved.

Remark 5. Let us note that for every $I\in BC(S)$ the space $X^*(I)$ is topologically conjugated to $X(I)$, but X^*_{loc} is not topologically conjugated to X^{loc}.

For every $n \geq 1$ and $I \in BC(S)$ we consider the Banach spaces

$$X_n(I) = L_{q_0}(I;H_n) \subset X(I), \quad X^*_n(I) = L_{p_0}(I;H_n) \subset X^*(I),$$

where $p_0 := \max\{r_1, r_2\}$, $q_0^{-1} + p_0^{-1} = 1$ with the natural norms. The space $L_{q_0}(I;H_n)$ is isometrically isomorphic to $X^*_n(I)$, the conjugate space of $X_n(I)$, moreover, the map

$$X_n(I)\times X^*_n(I) \ni (f,x) \to \int_I (f(\tau),x(\tau))_{H_n}\,d\tau = \int_I (f(\tau),x(\tau))\,d\tau = \langle f,x\rangle_{X_n(I)}$$

is the duality form on $X_n(I)\times X^*_n(I)$. This statement is correct in virtue of

$$L_{q_0}(I;H_n) \subset L_{q_0}(I;H) \subset L_{r_1'}(I;H) + L_{r_2'}(I;H) + L_{q_1}(I;V_1^*) + L_{q_2}(I;V_2^*).$$

Let us point out that $\langle\cdot,\cdot\rangle_I \mid_{X_n(I)\times X^*_n(I)} = \langle\cdot,\cdot\rangle_{X_n(I)}$.

Let us also consider the space

$$X_n^{\text{loc}} = \left\{y : S \to H_n \mid \forall\,\alpha\in\Theta \quad y|_{I_\alpha} \in X_n(I_\alpha)\right\},$$

which topology is compatible with the set of seminorms $\{\|\cdot\|_{X_n(I_\alpha)}\}_{\alpha\in\Theta}$, and

$$X^*_{n\mathrm{loc}} = \{y : S \to H_n \mid \forall\alpha\in\Theta \quad y|_{I_\alpha} \in X^*_n(I_\alpha)\},$$

which topology is compatible with the set of seminorms $\{\|\cdot\|_{X^*_n(I_\alpha)}\}_{\alpha\in\Theta}$.

Proposition 1. For every $n \geq 1$ we have $X_n^{\mathrm{loc}} = P_n X^{\mathrm{loc}}$, i.e.

$$X_n^{\mathrm{loc}} = \{P_n f(\cdot) \mid f(\cdot) \in X^{\mathrm{loc}}\}$$

Moreover, if the triple $(\{H_j\}_{j\geq 1}; V_i; H)$, $i = 1,2$ satisfies the condition (γ) with $C = C_i$, then for each $f \in X_{\mathrm{loc}}$, $n \geq 1$ and $I \in BC(S)$ it results in $\|P_n f\|_{X(I)} \leq$ $\leq \max\{C_1, C_2\}\|f\|_{X(I)}$.

Proof. To prove this proposition we will use [7] (proposition 3). Now we consider the first part.

"$\subset$" Let $x \in X_n^{\mathrm{loc}}$ be arbitrary fixed. Then for almost all $t \in S$ $P_n x(t) = x(t)$. Moreover, for every $I \in BC(S)$ $x|_I \in X_n(I) \subset X(I)$. Thus, $x \in P_n X^{\mathrm{loc}}$.

"$\supset$" Let $x \in P_n X^{\mathrm{loc}}$ be arbitrary fixed. Then for some $y \in X^{\mathrm{loc}}$ $P_n y(t) = x(t)$ for almost all $t \in S$. In virtue of [7](proposition 3) and the definition of X^{loc} it follows that for every $I \in BC(S)$ $x|_I = P_n y|_I \in X_n(I)$. Thus, $x \in X_n^{\mathrm{loc}}$.

The second part of the given proposition is the direct corollary of [7] (proposition 3). This completes the proof.

Proposition 2. For every $n \geq 1$ we have $X^*_{n\mathrm{loc}} = P_n X^*_{\mathrm{loc}}$, i.e. $X^*_{n\mathrm{loc}} = \{P_n y(\cdot) \mid y(\cdot) \in X^*_{\mathrm{loc}}\}$, and $\langle f, P_n y\rangle_I = \langle f, y\rangle_I \quad \forall I \in BC(S)$, $y \in X^*_{\mathrm{loc}}$, $f \in X_n^{\mathrm{loc}}$. Furthermore, if the triple $(\{H_j\}_{j\geq 1}; V_i; H)$, $i = 1,2$ satisfies condition (γ) with $C = C_i$, then it results in $\|P_n y\|_{X^*(I)} \leq \max\{C_1, C_2\}\|y\|_{X^*(I)}$ $\forall I \in BC(S)$, $y \in X^*_{\mathrm{loc}}$ and $n \geq 1$

Proof. To prove this proposition we use [7] (proposition 4). Now we consider the first part.

"$\subset$" Let $f \in X^*_{n\mathrm{loc}}$ be arbitrary fixed. Then for almost all $t \in S$ $P_n f(t) = f(t)$. Moreover, for every $I \in BC(S)$ $f|_I \in X^*_n(I) \subset X^*(I)$. Thus, $f \in P_n X^*_{\mathrm{loc}}$.

"$\supset$" Let $f \in P_n X^*_{\mathrm{loc}}$ be arbitrary fixed. Then for some $g \in X^*_{\mathrm{loc}}$ $P_n g(t) = g(t)$ for almost all $t \in S$. In virtue of [7](proposition 4) and the definition of X^*_{loc} it follows that for every $I \in BC(S)$ $f|_I = P_n f|_I \in X^*_n(I)$. Thus, $x \in X^*_{n\,\mathrm{loc}}$.

The last statements of the proposition is direct corollary of [7] (proposition 4).

Proposition is proved.

Proposition 3 Under the condition $\max\{r_1, r_2\} < +\infty$, the set $\bigcup_{n\ge 1} X^*_{n\,\mathrm{loc}}$ is dense in X^*_{loc}.

Proof. Arguing by contradiction, let us assume that for some $f \in X^*_{\mathrm{loc}}$ there is an open set from the base of topology of the locally convex linear topological space X^*_{loc}

$$\mathcal{O} = \bigcap_{k=1}^{n} V(\alpha_k, \varepsilon_k),$$

where $n \ge 1$ $\varepsilon_k > 0$, $\alpha_k \in \Theta$, $k = \overline{1,n}$,

$$V(\alpha, \varepsilon) = \left\{ u \in X^*_{\mathrm{loc}} \mid \|u\|_{X^*(I_\alpha)} < \varepsilon \right\}, \quad \alpha \in \Theta, \ \ \varepsilon > 0,$$

such that

$$\left(\bigcup_{n\ge 1} X^*_{n\,\mathrm{loc}} \right) \bigcap (f + \mathcal{O}) = \varnothing.$$

Thus

$$\left(\bigcup_{n\ge 1} X^*_{n\,\mathrm{loc}} \right) \bigcap (f + \mathcal{O}) \supset \left(\bigcup_{n\ge 1} X^*_{n\,\mathrm{loc}} \right) \bigcap (f + V(\alpha_0, \varepsilon_0)) = \varnothing, \tag{7}$$

where $\varepsilon_0 = \dfrac{1}{n} \min_{k=\overline{1,n}} \varepsilon_k > 0$, $\alpha_0 \in \Theta(\mathbb{R}) : BC(\mathbb{R}) \ni I_{\alpha_0} \supset \bigcup_{k=\overline{1,n}} I_{\alpha_k}$. Because of the set

$$\left(f + V(\alpha_0, \varepsilon_0)\right)|_{I_{\alpha_0}} = \left\{ f|_{I_{\alpha_0}} + g|_{I_{\alpha_0}} \mid g \in V(\alpha_0, \varepsilon_0) \right\}$$

is open in $X^*(I_{\alpha_0})$, due to [7](proposition 5) the set

$$\bigcup_{n\ge 1} \left(X^*_{n\,\mathrm{loc}}|_{I_{\alpha_0}} \right) = \bigcup_{n\ge 1} X^*_n(I_{\alpha_0}) \text{ is dense in } \left(X^*(I_{\alpha_0}), \|\cdot\|_{X^*(I_{\alpha_0})} \right)$$

and from (7) we obtain the contradiction.

The proof is concluded.

Now for an arbitrary $I \in BC(S)$ we consider Banach space

$$W^*(I) = \{ y \in X^*(I) \mid y' \in X(I) \}$$

with the norm $\|y\|_{W(I)^*} = \|y\|_{X(I)^*} + \|y'\|_{X(I)}$, where we mean the derivative y' of an element $y \in X(I)^*$ in the sense of the scalar distribution space $\mathcal{D}^*(I;V^*) = \mathcal{L}(\mathcal{D}(I);V_w^*)$, where V_w^* is equal to V^* with topology $\sigma(V^*;V)$ [10].

Together with $W(I)^*$ we consider the Banach space

$$W_i^*(I) = \{y \in L_{p_i}(I;V_i) \,|\, y' \in X(I)\}, \quad i = 1,2,$$

with the norm

$$\|y\|_{W_i^*(I)} = \|y\|_{L_{p_i}(I;V_i)} + \|y'\|_{X(I)} \quad \forall\, y \in W_i^*(I).$$

Also we consider the space $W_0^*(I) = W_1^*(I) \cap W_2^*(I)$ with the norm

$$\|y\|_{W_0^*(I)} = \|y\|_{L_{p_1}(I;V_1)} + \|y\|_{L_{p_2}(I;V_2)} + \|y'\|_{X(I)} \quad \forall y \in W_0^*(I).$$

Notice that the space $W^*(I)$ is continuously embedded in $W_i^*(I)$ for $i = \overline{0,2}$.

Let us set $W_{0\mathrm{loc}}^* = \{y \in L_{p_1}^{\mathrm{loc}}(S;V_1) \cap L_{p_2}^{\mathrm{loc}}(S;V_2) | y' \in X^{\mathrm{loc}}\}$, where the derivative y' of an element $y \in L_{p_1}^{\mathrm{loc}}(S;V_1) \cap L_{p_2}^{\mathrm{loc}}(S;V_2)$ is regarded in the sense of space of distributions $\mathcal{D}^*(S;V^*)$ and in this space a subbase of topology σ is assigned through the following sets:

$$\mathcal{C} = \left\{ U(\alpha,\varepsilon) = \left\{ u \in W_{0\mathrm{loc}}^* \;\middle|\; \|u\|_{L_{p_1}(I_\alpha;V_1)} + \|u\|_{L_{p_2}(I_\alpha;V_2)} + \right.\right.$$

$$\left.\left. + \|u'\|_{X(I_\alpha)} < \varepsilon \right\} \;\middle|\; \alpha \in \Theta, \varepsilon > 0 \right\}.$$

Lemma 3. $\mathcal{C}$ is a subbase of some topology σ in $W_{0\mathrm{loc}}^*$, which turns the given space into separable locally convex linear topological space and, moreover:

a) σ is compatible with the set of seminorms

$$\{\rho_\alpha(u) = \|u\|_{L_{p_1}(I_\alpha;V_1)} + \|u\|_{L_{p_2}(I_\alpha;V_2)} + \|u'\|_{X(I_\alpha)}\}_{\alpha \in \Theta},$$

divides points on $W_{0\mathrm{loc}}^*$;

b) a set $E \subset W_{0\mathrm{loc}}^*$ is bounded only when for every $\alpha \in \Theta$ ρ_α is bounded on E.

Proof. As well as in lemma 1 it is enough to show that system of seminorms $\{\rho_\alpha\}_{\alpha \in \Theta}$ divides points on $W_{0\mathrm{loc}}^*$. Let $u \in W_{0\mathrm{loc}}^* \setminus \{\bar{0}\}$, then $\lambda(t \in S \,|\, u(t) \neq \bar{0}) > 0$. Because of S is a subset of a real line which can be presented as no more than numerable join of convex sets in $\mathbb{R}$, we have

$\exists \alpha_0 \in \Theta$: $\|u\|_{W_0^*(I_{\alpha_0})} > 0$. From here it follows $\rho_{\alpha_0}(u) > 0$, as it was to be shown.

From [12, theorem 1.37] it follows that the system of seminorms $\{\rho_\alpha\}_{\alpha\in\Theta}$ generates some locally convex topology σ on $W_{0\text{loc}}^*$, which converts the given space in locally convex linear topological space, whose local base we obtain by final intersections of such sets:

$$\left\{V(I_\alpha, \varepsilon) = \{u \in W_{0\text{loc}}^* \mid \rho_\alpha(u) < \varepsilon\} \mid \alpha \in \Theta,\ \varepsilon > 0\right\}.$$

The statement b) follows from the same theorem.

The lemma is proved.

For a subset of a real line S which can be presented as no more than numerable join of convex sets in $\mathbb{R}$, distinct from a point, let us denote by $BCC(S) = \{I_\alpha\}_{\alpha\in\Delta}$ the family of all convex compact sets from S, distinct from a point. We notice that the family of all subset of a real line S which can be presented as no more than numerable join of convex sets in $\mathbb{R}$, distinct from a point, coincides with the family of all subset of a real line S which can be presented as no more than numerable join of convex compact sets in $\mathbb{R}$, distinct from a point.

Let us also consider the space

$$C^{\text{loc}}(S; H) = \left\{y : S \to H \mid \forall \alpha \in \Delta \quad y|_{I_\alpha} \in C(I_\alpha; H)\right\}$$

which topology is compatible with the set of seminorms $\{\|\cdot\|_{C(I_\alpha;H)}\}_{\alpha\in\Delta}$.

Theorem 2. It results in $W_{0\text{loc}}^* \subset C^{\text{loc}}(S; H)$ with continuous embedding. Moreover, for every $y, \xi \in W_0^*$ and $s, t \in S$: $s < t$ and $(s,t) \in BC(S)$, the next formula of integration by parts takes place

$$(y(t), \xi(t)) - (y(s), \xi(s)) = \int_s^t \{(y'(\tau), \xi(\tau)) + (y(\tau), \xi'(\tau))\} d\tau. \tag{8}$$

In particular, when $y = \xi$ we have:

$$\frac{1}{2}(\|y(t)\|_H^2 - \|y(s)\|_H^2) = \int_s^t (y'(\tau), y(\tau)) d\tau\,.$$

Proof. At first let us prove the embedding $W_{0\text{loc}}^* \subset C^{\text{loc}}(S; H)$ in the sense of the set theory. Let $y \in W_{0\text{loc}}^*$ be fixed. Then for every $t \in S$, due to the set S can be presented as no more than numerable join of convex compact sets in $\mathbb{R}$, distinct from a point, there is $I \in BCC(S)$ such that $t \in I$. Moreover, we can consider that t is an interior point of I in the space $(S, |\cdot|)$. Hence, due to the definition of $W_{0\text{loc}}^*$ and [7, theorem 5] it follows that $y|_I \in W_0^*(I) \subset C(I; H)$.

Thus the function $y: S \to H$ is continuous in the point t. The necessary statement follows from the arbitrary of $t \in S$.

Now let us prove the continuous embedding $W^*_{0\text{loc}} \subset C^{\text{loc}}(S;H)$. Since the set S can be presented as no more than numerable join of convex compact sets in $\mathbb{R}$, distinct from a point, there exists $\Xi \subset \Delta$ ($\text{card}\,\Xi \le \aleph_0$) such that $\bigcup_{\alpha \in \Xi} I_\alpha = S$. So, it is enough to show that for every $\alpha \in \Xi$ there is a continuous seminorm $\mu_\alpha : C^{\text{loc}}(S;H) \to \mathbb{R}$ and a constant $C_\alpha > 0$ such that

$$\|y\|_{W_0^*(I_\alpha)} \le C_\alpha \mu_\alpha(u) \quad \forall u \in W^*_{0\text{loc}}.$$

This fact follows from [7](theorem 5) because of for every $\alpha \in \Xi$ $I_\alpha \in BCC(S)$.

At last we obtain formula (8) by using [7] (theorem 5) with $S = [s,t]$.

The theorem is proved.

Let us consider the space $W^*_{\text{loc}} = \{y \in X^*_{loc} | y' \in X^{\text{loc}}\}$, which topology is compatible with the set of seminorms $\{\|\cdot\|_{W^*(I_\alpha)}\}_{\alpha \in \Theta}$.

In virtue of $W^*_{\text{loc}} \subset W^*_{0\text{loc}}$ with continuous embedding and due to the latter theorem the next statement is true.

Corollary 1. $W^*_{\text{loc}} \subset C^{\text{loc}}(S;H)$ with continuous embedding. Moreover, for every $y, \xi \in W_0^*$ and $s,t \in S$: $s < t$ and $(s,t) \in BC(S)$, formula (8) takes place.

For every $n \ge 1$ and $I \in BC(S)$ let us introduce the Banach space

$$W_n^*(I) = \left\{ y \in X_n^*(I) \mid y' \in X_n(I) \right\}$$

with the norm $\|y\|_{W_n^*(I)} = \|y\|_{X_n^*(I)} + \|y'\|_{X_n(I)}$, where the derivative y' is considered in sense of scalar distributions space $\mathcal{D}^*(I;H_n)$ and the space

$$W^*_{n\text{loc}} = \{y \in X^*_{n\text{loc}} | y' \in X_n^{\text{loc}}\},$$

which topology is compatible with the set of seminorms $\{\|\cdot\|_{W_n^*(I_\alpha)}\}_{\alpha \in \Theta}$.

As far as $\mathcal{D}^*(S;H_n) = \mathcal{L}(\mathcal{D}(S);H_n) \subset \mathcal{L}(\mathcal{D}(S);V_\omega^*) = \mathcal{D}^*(S;V^*)$ it is possible to consider the derivative of an element $y \in X_n^*(S)$ in the sense of $\mathcal{D}^*(S;V^*)$. Notice that for every $n \ge 1$ $W^*_{n\text{loc}} \subset W^*_{n+1\text{loc}} \subset W^*_{\text{loc}}$.

Proposition 4. For every $y \in X^*_{\text{loc}}$ and $n \ge 1$ it results in $P_n y' = (P_n y)'$, where we mean the derivative of an element of X^*_{loc} in the sense of the scalar distributions space $\mathcal{D}^*(S;V^*)$.

Remark 6. We point out that in virtue of the previous assumptions the derivatives of an element of $X^*_{n\,\mathrm{loc}}$ in the sense of $\mathcal{D}(S;V^*)$ and in the sense of $\mathcal{D}(S;H_n)$ coincide.

Proof. It is sufficient to show that for every $\varphi \in \mathcal{D}(S)$ $P_n y'(\varphi) = (P_n y)'(\varphi)$. In virtue of definition of derivative in sense of $\mathcal{D}^*(S;V^*)$ we have

$$\forall \varphi \in D(S) \quad P_n y'(\varphi) = -P_n y(\varphi') = -P_n \int_S y(\tau)\varphi'(\tau)d\tau =$$

$$= -\int_S P_n y(\tau)\varphi'(\tau)d\tau = -(P_n y)(\varphi') = (P_n y)'(\varphi).$$

The proposition is proved.

From the propositions 2, 1, 4 it follows the next

Proposition 5. For every $n \ge 1$ $W^*_{n\mathrm{loc}} = P_n W^*_{\mathrm{loc}}$, i.e.

$$W^*_{n\,\mathrm{loc}} = \{P_n y(\cdot) \mid y(\cdot) \in W^*_{l\mathrm{loc}}\}.$$

Moreover, if the triple $(\{H_i\}_{i\ge1}; V_j; H)$, $j = 1,2$ satisfies condition (γ) with $C = C_j$, then for every $y \in W^*_{\mathrm{loc}}$, $n \ge 1$ and $\alpha \in \Theta$ it results in $\|P_n y(\cdot)\|_{W^*(I_\alpha)} \le \max\{C_1, C_2\}\|y(\cdot)\|_{W^*(I_\alpha)}$.

Theorem 3. Let the triple $(\{H_i\}_{i\ge1}; V_j; H)$, $j = 1,2$ satisfy condition (γ) with $C = C_j$. Moreover, let $D \subset X^*_{\mathrm{loc}}$ be bounded in X^*_{loc} set and $E \subset X_{\mathrm{loc}}$ bounded in X_{loc}. For every $n \ge 1$ let us consider $D_n := \{y_n \in \in X^*_{n\,\mathrm{loc}} | y_n \in D$ and $y'_n \in P_n E\} \subset W^*_{n\,\mathrm{loc}}$ Then for each $\alpha \in \Theta$, $n \ge 1$ and $y_n \in D_n$

$$\|y_n\|_{W^*(I_\alpha)} \le \|D\|^\alpha_+ + C \cdot \|E\|^\alpha_+, \tag{9}$$

where $C = \max\{C_1, C_2\}$, $\|D\|^\alpha_+ = \sup\limits_{y\in D} \|y\|_{X^*(I_\alpha)}$ and $\|E\|^\alpha_+ = \sup\limits_{f\in E} \|f\|_{X(I_\alpha)}$, i.e. the set

$$\bigcup_{n\ge1} D_n \quad \text{is bounded in} \quad W^*_{\mathrm{loc}}$$

and, consequently, bounded in $C^{\mathrm{loc}}(S;H)$.

Remark 7. Due to proposition 1 D_n is well-defined and $D_n \subset W^*_{n\,\mathrm{loc}}$.

Remark 8. A priori estimates (like (9)) appear at studying of global solvability of differential–operator equations, inclusions and evolutional variational inequalities in nonreflexive Banach and Frechet spaces with maps of w_λ-pseudomonotone type by using Faedo–Galerkin method (see [1, 2]) at boundary transition, when it is necessary to obtain a priori estimates of approximate solutions y_n in X^*_{loc} and its derivatives y'_n in X^{loc}.

Proof. The assertion of the theorem is immediate consequence of the inequality

$$\forall n \geq 1,\ \alpha \in \Theta,\ y_n \in D_n \quad \|y_n\|_{W^*(I_\alpha)} = \|y_n\|_{X^*(I_\alpha)} + \|y'_n\|_{X(I_\alpha)} \leq$$

$$\leq \|D\|_+^\alpha + \|P_n E\|_+^\alpha \leq \|D\|_+^\alpha + \max\{C_1, C_2\}\|E\|_+^\alpha,$$

that is valid in virtue of proposition 1.

Further, let B_0, B_1, B_2 be some Banach spaces such that

$$B_0, B_2 \text{ are reflexive, } B_0 \subset B_1 \text{ with compact embedding} \tag{10}$$

$$B_0 \subset B_1 \subset B_2 \text{ with continuous embedding} \tag{11}$$

$p_0, p_1 \in [1; +\infty)$ be arbitrary numbers. For every $\alpha \in BC(S)$ we consider the set with the natural operations $W(I_\alpha) = \{v \in L_{p_0}(I_\alpha; B_0) | v' \in L_{p_1}(I_\alpha; B_2)\}$, where the derivative v' of an element $v \in L_{p_0}(I_\alpha; B_0)$ is considered in the sense of the scalar distribution space $\mathcal{D}(I_\alpha; B_2)$. It is obvious that $W(I_\alpha) \subset L_{p_0}(I_\alpha; B_0)$. Let us also consider the set

$$W^{\text{loc}} = \{y \in L^{\text{loc}}_{p_0}(S; B_0) | y' \in L^{\text{loc}}_{p_1}(S; B_2)\}.$$

It is clear, that $W^{\text{loc}} \subset L^{\text{loc}}_{p_0}(S; B_0)$.

Theorem 4. W^{loc} with the natural operations, which is topologically compatible with set of seminorms $\{\rho_\alpha(\cdot) = \|\cdot\|_{W(I_\alpha)}\}_{\alpha \in \Theta}$ is a Frechet space.

Proof. Since the set S can be presented as no more than numerable join of convex sets in $\mathbb{R}$ there exists $\Xi \subset \Theta$ ($\operatorname{card} \Xi \leq \aleph_0$) such that

$$S = \bigcup_{\alpha \in \Xi} I_\alpha.$$

Thus, as well as in lemma 3, we can prove that the no more than numerable system of seminorms $\{\rho_\alpha(\cdot)\}_{\alpha \in \Xi}$ divides points on W^{loc}. Thus, the families of seminorms $\{\rho_\alpha(\cdot)\}_{\alpha \in \Xi}$ and $\{\rho_\alpha(\cdot)\}_{\alpha \in \Theta}$ are equivalent and the locally convex linear topological space $\left(W^{\text{loc}}, \{\rho_\alpha(\cdot)\}_{\alpha \in \Xi}\right)$ is metrizable.

Now let us prove that the metrizable space W^{loc} is complete. Let us consider a Cauchy sequence $\{y_n\}_{n \geq 1} \subset W^{\text{loc}}$; without loss of generality we can assume that for every $\alpha, \beta \in \Xi$: $\alpha \neq \beta$ it follows that $I_\alpha \cap I_\beta = \varnothing$. We also consider

$$\Xi = \{\alpha_1 < \alpha_2 < \ldots < \alpha_n < \alpha_{n+1} < \ldots\}.$$

i_1) Because of $\{y_n\}_{n\ge1}\subset W^{\mathrm{loc}}$ is a Cauchy sequence also $\left\{y_n|_{I_{\alpha_1}}\right\}_{n\ge1}$ is a Cauchy sequence in $W(I_{\alpha_1})$. Thus in virtue of [7] (theorem 8) there is a subsequence $\{v_{1,n}\}_{n\ge1}$ of $\{y_n\}_{n\ge1}$ that converges in $W(I_{\alpha_1})$ to some $x_1\in W(I_{\alpha_1})$;

i_2) analogously to i_1), due to $\{v_{1,n}\}_{n\ge1}\subset W^{\mathrm{loc}}$ is a Cauchy sequence the same follows for $\left\{v_{1,n}|_{I_{\alpha_2}}\right\}_{n\ge1}$ in $W(I_{\alpha_2})$. Thus there is a subsequence $\{v_{2,n}\}_{n\ge1}$ of $\{v_{1,n}\}_{n\ge1}$ that converges in $W(I_{\alpha_2})$ to some $x_2\in W(I_{\alpha_2})$;

i_m) due to $\{v_{m,n}\}_{n\ge1}\subset W^{\mathrm{loc}}$ is a Cauchy sequence the same follows for $\left\{v_{m,n}|_{I_{\alpha_m}}\right\}_{n\ge1}$ in $W(I_{\alpha_m})$. Thus there is a subsequence $\{v_{m+1,n}\}_{n\ge1}$ of $\{v_{m,n}\}_{n\ge1}$ that converges in $W(I_{\alpha_m})$ to some $x_m\in W(I_{\alpha_m})$.

Thanks to $i_1), i_2), \ldots$, using the diagonal Cantor method, we can choose a subsequence $\{y_{n_k}\}_{k\ge1}=\{v_{n,n}\}_{n\ge1}$ from $\{y_n\}_{n\ge1}$ that converges in $W(I_{\alpha_m})$ to $x_m\in W(I_{\alpha_m})$ for every $m\ge1$.

By setting $y(t)=x_m(t)$, $t\in I_{\alpha_m}$, $m\ge1$ we obtain that for every $\alpha\in\Xi$ $\rho_\alpha(y_{n_k}-y)\to0$ as $k\to\infty$.

To conclude the proof we remark that $y\in W^{\mathrm{loc}}$ in virtue of the definition W^{loc} and the condition: $\forall\,\alpha,\beta\in\Xi$: $\alpha\ne\beta$ it follows that $I_\alpha\cap I_\beta=\varnothing$.

The theorem is proved.

Analogously with the proof of theorem 4 we can obtain the next:

Theorem 5. The set W^*_{loc} (respectively $W^*_{i\,\mathrm{loc}}$, $i=\overline{0,2}$) with the natural operations, which topology is compatible with the set of seminorms $\{\|\cdot\|_{W^*(I_\alpha)}\}_{\alpha\in\Theta}$ (respectively $\{\|\cdot\|_{W_i^*(I_\alpha)}\}_{\alpha\in\Theta}$, $i=\overline{0,2}$) is a Frechet space.

Theorem 6. Under conditions (10)–(11), we have $W^{\mathrm{loc}}\subset C^{\mathrm{loc}}(S;B_2)$ with the continuous embedding.

Proof. At first let us prove the embedding $W^{\mathrm{loc}}\subset C^{\mathrm{loc}}(S;B_2)$ in the sense of the set theory. Let $y\in W^{\mathrm{loc}}$ be fixed. Then for every $t\in S$, since the set S can be presented as no more than numerable join of convex compact sets in $\mathbb{R}$, distinct from a point, there is $I\in BCC(S)$ such that $t\in I$. Moreover, we can consider that t is an interior point of I in the space $(S,|\cdot|)$. Hence, due to the definition of W^{loc} and [7, theorem 5] it follows that $y|_I\in W(I)\subset C(I;B_2)$. Thus the function $y:S\to B_2$ is continuous in the point t.

Now let us prove the continuous embedding $W^{\text{loc}} \subset C^{\text{loc}}(S;B_2)$. Since the set S can be presented as no more than numerable join of convex compact sets in $\mathbb{R}$, distinct from a point, there exists $\Xi \subset \Delta$ ($\operatorname{card}\Xi \le \aleph_0$) such that $\bigcup\limits_{\alpha\in\Xi} I_\alpha = S$.

So, it is enough to show that for every $\alpha \in \Xi$ there is a continuous seminorm $\mu_\alpha : C^{\text{loc}}(S;B_2) \to \mathbb{R}$ and a constant $C_\alpha > 0$ such that

$$\|y\|_{W_0^*(I_\alpha)} \le C_\alpha \mu_\alpha(u) \quad \forall u \in W^{\text{loc}}.$$

In fact for every $\alpha \in \Xi$ $I_\alpha \in BCC(S)$. Thus the above inequality is true in virtue of [7, theorem 9].

The theorem is proved.

The next result represents a generalization of the classical compactness lemma [11] (theorem 1.5.1, p.70) into the case $p_0, p_1 \in [1;+\infty)$.

Theorem 7. [7, theorem 10] Under conditions (10)–(11), for every $p_0, p_1 \in [1;+\infty)$ the Banach space W is compactly embedded in $L_{p_0}(S;B_1)$.

The proof follows from the next lemmas:

Lemma 4. [7, lemma 3] For every $y \in W$ and $h \in \mathbb{R}$ it results in $\|y - y_h\|_{L_1(S;B_2)} \le h\|y'\|_{L_1(S;B_2)}$, where $y_h(t) = \begin{cases} y(t+h), & \text{if } t+h \in S, \\ \bar{0}, & \text{otherwice.} \end{cases}$

Lemma 5. [7, theorem 7] Let conditions (10)–(11) for B_0, B_1, B_2 are valid, $p_0, p_1 \in [1;+\infty)$, S a finite time interval and $K \subset L_{p_1}(S;B_0)$ such that

$a)$ K is bounded in $L_{p_1}(S;B_0)$;

b) for all $\varepsilon > 0$ there exists $\delta > 0$ such that from $0 < h < \delta$ it results in

$$\int\limits_S \|u(\tau) - u(\tau+h)\|_{B_2}^{p_0} d\tau < \varepsilon \quad \forall u \in K.$$

Then K is precompact in $L_{\min\{p_0;p_1\}}(S;B_1)$.

Furthermore, if for some $q > 1$ K is bounded in $L_q(S;B_1)$, then K is precompact in $L_p(S;B_1)$ for every $p \in [1,q)$.

Lemma 6. [7, corollary 2] Let assumptions (10)–(11) for the Banach spaces B_0, B_1 and B_2 are valid, $p_1 \in [1;+\infty]$, $S = [0,T]$ and the set $K \subset L_{p_1}(S;V)$ such that

a) K is precompact set in $L_{p_1}(S;B_2)$;

b) K is bounded set in $L_{p_1}(S;B_0)$.

Then K is precompact set in $L_{p_1}(S;B_1)$.

The next result is a generalization of the compactness lemma [8] (theorem 2) into the case $p_0, p_1 \in [1;+\infty)$.

Theorem 8. Under above assumptions, the embedding W^{loc} in $L_{p_0}^{\text{loc}}(S;B_1)$ is compact, that is, an arbitrary bounded in W^{loc} set is precompact in $L_{p_0}^{\text{loc}}(S;B_1)$.

Proof. Arguing by contradiction, let $\{y_n\}_{n\ge1} \subset W^{\text{loc}}$ be bounded in W^{loc} sequence that has no any accumulation point in $L_{p_0}^{\text{loc}}(S;B_1)$. From [12, theorem 1.37] it follows, that for every convex bounded set $S_\alpha \subset S$

$$\sup_{n\ge1}\left(\|y_n\|_{L_{p_0}(S_\alpha;B_0)} + \|y_n'\|_{L_{p_1}(S_\alpha;B_2)}\right) < +\infty. \tag{12}$$

As on real line the arbitrary convex set can be presented as join no more, than numerable number of bounded convex sets. Without loss of generality we suppose $S = \bigcup_{\alpha\in\Xi} S_\alpha$, where S_α is bounded convex set in $\mathbb{R}$ $\forall \alpha \in \Xi$ and $\operatorname{card} \Xi \le \aleph_0$. Further we consider only those $\alpha \in \Xi$ for which $\lambda(S_\alpha) > 0$.

Let it be $\Xi = \{\alpha_n\}_{n\ge1}$, then:

i_1) from (12) and theorem 7 about compactness we obtain there is a subsequence $\{v_{1,n}\}_{n\ge1}$ of $\{y_n\}_{n\ge1}$, that is fundamental in the space $L_{p_0}(S_{\alpha_1};B_1)$;

i_2) analogously to i_1), from (12) and theorem 7 about compactness it follows, there exists $\{v_{2,n}\}_{n\ge1} \subset \{v_{1,n}\}_{n\ge1}$ that is fundamental in $L_{p_0}(S_{\alpha_2};B_1)$;

i_m) from (12) and theorem 7 about compactness it follows, there exists $\{v_{m,n}\}_{n\ge1} \subset \{v_{m-1,n}\}_{n\ge1}$, that is fundamental in $L_{p_0}(S_{\alpha_m};B_1)$;

Thanks to i_1), i_2),…, using the diagonal Cantor method, we can choose a subsequence $\{y_{n_k}\}_{k\ge1} = \{v_{n,n}\}_{n\ge1}$ from $\{y_n\}_{n\ge1}$ that is fundamental in $L_{p_0}^{\text{loc}}(S;B_1)$. This is a contradiction.

The theorem is proved.

By the analogy with the last theorem, due to the lemma 6, we can obtain the next:

Theorem 9. Let assumptions (10)–(11) for the Banach spaces B_0, B_1 and B_2 are valid, $p_1 \in [1;+\infty)$, $S = [0,T]$ and the set $K \subset L_{p_1}^{\text{loc}}(S;V)$ such that

a) K is precompact set in $L_{p_1}^{\text{loc}}(S;B_2)$;

b) K is bounded set in $L_{p_1}^{\text{loc}}(S;B_0)$.

Then K is precompact set in $L_{p_1}^{\mathrm{loc}}(S;B_1)$.

Now we combine all results to obtain the necessary a priori estimates.

Theorem 10. Let all conditions of theorem 3 are satisfied and $V \subset H$ with compact embedding. Then estimate (9) is true and the set

$$\bigcup_{n\geq 1} D_n \quad \text{is bounded in} \quad C^{\mathrm{loc}}(S;H) \quad \text{and precompact in} \quad L_p^{\mathrm{loc}}(S;H)$$

for every $p \geq 1$.

Proof. Estimation (9) follows from theorem 3. Now we apply the compactness theorem 8 with $B_0 = V$, $B_1 = H$, $B_2 = V^*$, $p_0 = 1$, $p_1 = 1$. Notice that $X_{\mathrm{loc}}^* \subset L_1^{\mathrm{loc}}(S;V)$ and $X^{\mathrm{loc}} \subset L_1(S;V^*)$ with continuous embedding. Hence, the set

$$\bigcup_{n\geq 1} D_n \quad \text{is precompact in} \quad L_1^{\mathrm{loc}}(S;H).$$

In virtue of (9) and of theorem 2 on continuous embedding W_{loc}^* in $C^{\mathrm{loc}}(S;H)$ it follows that the set

$$\bigcup_{n\geq 1} D_n \quad \text{is bounded in} \quad C^{\mathrm{loc}}(S;H).$$

Further, we complete the proof by using standard conclusions, Lebesgue theorem and the diagonal Cantor method.

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