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THE CLASSES AND THE MAIN PROPERTIES OF THE MULTI-VALUED W_{λ_0}-PSEUDOMONOTONE MAPS

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We consider the main classes of W_{λ_0}-pseudomonotone multi-valued maps. The main properties of these operators have been investigated. The new classes of these operators have been obtained.

1. INTRODUCTION

One of the most effective approach to investigate nonlinear problems, represented by partial differential equations, inclusions and evolution inequalities with boundary values, consists in the reduction of them into equations in Banach spaces governed by nonlinear operators. The given theory was developed by many authors [1–20]. In particular, the idea of W_{λ_0}-pseudomonotone maps was introduced in I.V.Skripnik paper and it was developed in papers [3–10, 14, 17, 19].

Here we investigate the main properties of the given multi-valued operators. In particular, we will show that the sum of these operators is W_{λ_0}-pseudomonotone, it is difficult in the classical definitions. We will also consider the generous pseudomonotone operators and we will prove its W_{λ_0}-pseudomonotony.

Finally, we will consider a class of the W_{λ_0}-pseudomonotone multi-valued maps and, by using the obtained results and one abstract result for such operators [7], we obtain the solvability for a class of nonlinear evolutional problems.

2. CLASSES OF MAPS

Let $(Y, \|\cdot\|_Y)$ be some Banach space, $A: Y \to 2^{Y^*}$ be a multi-valued map. We consider its corresponding maps $\operatorname{co} A: Y \to 2^{Y^*}$ and $\overline{\operatorname{co}}^{*} A: Y \to 2^{Y^*}$ defined by the relations $(\operatorname{co} A)(y) = \operatorname{co}(A(y))$ and $(\overline{\operatorname{co}}^{*} A(y)) = \overline{\operatorname{co}}^{*}(A(y))$ respectively, where $\overset{*}{\_}$ is $*$-weak closure in the space Y^*.

For each multi-valued map A we introduce its *upper* and *lower function of support*:

$$[A(y),\omega]_+ = \sup_{d\in A(y)} < d, w >_X, \quad [A(y),\omega]_- = \inf_{d\in A(y)} < d, w >_X,$$

where $y,\ \omega \in X$. We also consider its *upper* and *lower* norms:

$$\| A(y) \|_+ = \sup_{d\in A(y)} \| d \|_{X^*}, \quad \| A(y) \|_- = \inf_{d\in A(y)} \| d \|_{X^*} .$$

Proposition 1. Let $A, B : Y \to 2^{Y^*}$. Then the next relations are valid:

1) $[A(y), v_1 + v_2]_+ \le [A(y), v_1]_+ + [A(y), v_2]_+$,

$[A(y), v_1 + v_2]_- \ge [A(y), v_1]_- + [A(y), v_2]_-$,

$[A(y), v_1 + v_2]_+ \ge [A(y), v_1]_+ + [A(y), v_2]_-$,

$[A(y), v_1 + v_2]_- \le [A(y), v_1]_+ + [A(y), v_2]_- \quad \forall\, y,\ v_1,\ v_2 \in Y$;

2) $[A(y), v]_+ = -[A(y), -v]_-$,

$[A(y) + B(y), v]_{+(-)} = [A(y), v]_{+(-)} + [B(y), v]_{+(-)} \quad \forall\, y, v \in Y$;

3) $[A(y), v]_{+(-)} = [\overset{*}{\overline{\mathrm{co}}}\, A(y), v]_{+(-)} \quad \forall\, y, v \in Y$;

4) $[A(y), v]_{+(-)} \le \| A(y) \|_{+(-)} \| v \|_Y$,

$d_H(A(y), B(y)) \ge \left| \| A(y) \|_{+(-)} - \| B(y) \|_{+(-)} \right|$,

$\| A(y) - B(y) \|_+ \ge \left| \| A(y) \|_+ - B(y) \|_- \right|$,

where $d_H(\cdot,\cdot)$ is the Hausdorff metric;

5) $\| \overset{*}{\overline{\mathrm{co}}}\, A(y) \|_+ = \| A(y) \|_+$ and if the space Y is reflexive then

$$\| \overset{*}{\overline{\mathrm{co}}}\, A(y) \|_- = \| A(y) \|_- \quad \forall\, y \in Y;$$

6) the functional $\| \cdot \|_+ : C_v(X^*) \to \mathbf{R}_+$ defines the norm on $C_v(X^*)$;

7) the functional $\| \cdot \|_- : C_v(X^*) \to \mathbf{R}_+$ satisfies the conditions:

a) $\bar{0} \in A(y) \Leftrightarrow \| A(y) \|_- = 0$,

b) $\| \alpha A(y) \|_- = | \alpha | \| A(y) \|_- \quad \forall\, \alpha \in \mathbf{R},\ y \in X$,

c) $\| A(y) + B(y) \|_- \le \| A(y) \|_- + \| B(y) \|_-$.

Proof. The properties 1), 2), 4), 6), 7) can be proved directly. Property 3) is well known. Let us consider the property 5). It is obvious that $\| \overset{*}{\overline{\mathrm{co}}}\, A(y) \|_+ \ge$ $\ge \| \mathrm{co}\, A(y) \|_+ \ge \| A(y) \|_+$, and so we will prove the inverse inequality. For arbitrary $f \in \overset{*}{\overline{\mathrm{co}}}\, A(y)$ there exists the sequence $f_n \in \mathrm{co}\, A(y)$ such that $f_n \to f$ $*$-weakly in Y^* and from the Banach-Steinhaus theorem it follows

$$\| \mathrm{co}\, A(y) \|_+ \ge \varliminf_{n\to\infty} \| f_n \|_{Y^*} \ge \| f \|_{Y^*} .$$

Since the last inequality is valid for all $f \in \overline{\mathrm{co}}^* A(y)$ then

$$\| \mathrm{co}\, A(y) \|_+ = \| \overline{\mathrm{co}}^* \; A(y) \|_+ .$$

Let us prove that $\| \mathrm{co}\, A(y) \|_+ \le \| A(y) \|_+$. Let $f \in \mathrm{co}\, A(y)$ be arbitrary then for $n \ge 1$ there exist $\alpha_1, ..., \alpha_n$ ($\alpha_i \ge 0$, $\sum_{i=1}^{n} \alpha_i = 1$), $g_1, ..., g_n$ ($g_i \in A(y)$) such that $f = \sum_{i=1}^{n} \alpha_i g_i$. Hence

$$\| f \|_{Y^*} \le \sum_{i=1}^{n} \alpha_i \| g_i \|_{Y^*} \le \sum_{i=1}^{n} \alpha_i \| A(y) \|_+ = \| A(y) \|_+ .$$

From here and from the arbitrariness of $f \in \mathrm{co}\, A(y)$ we obtain the required inequality which proves the first equality in 5). Let us prove the second one. Let us introduce the mapping

$$f : (A(y) \subset Y^*) \times (\overline{B}_1 \subset Y) \to \mathbf{R}$$

defined by the equality $f(d, \xi) = < d, \xi >_Y$ where $\overline{B}_1$ is the unitary closed sphere in the space Y with the center in zero. Let $f_\xi(\cdot) = f(\cdot, \xi)$ then $f_\xi^*(p) = = [\overline{\mathrm{co}}^* A(y), p - \xi]_+$ and

$$\mathrm{dom}\, f_\xi^* = \{ p \in Y \,|\, [\overline{\mathrm{co}}^* A(y), p - \xi]_+ < +\infty \} .$$

Notice that $\overline{0} \in \mathrm{int}(\bigcup_{\xi \in B_1} \mathrm{dom}\, f_\xi^*)$. In fact, $\overline{0} \in \mathrm{dom}\, f_0^*$, $B_1 \subset \mathrm{dom}\, f_\xi^*$ and the function f satisfies the condition of non-symmetric theorem on minimax [14]. Therefore

$$\inf_{d \in A(y)} \sup_{\xi \in \overline{B}_1} f(d, \xi) = \sup_{\xi \in \overline{B}_1} \inf_{d \in A(y)} f(d, \xi),$$

from which the required equality follows.

Proposition 2. The inclusion $d \in \overline{\mathrm{co}}^* A(y)$ is fulfilled if and only if

$$[A(y), v]_+ \ge < d, v >_Y \quad \forall\, v \in Y .$$

Proof. Let $d \in \overline{\mathrm{co}}^* A(y)$ then $\forall\, v \in Y$, from the proposition 1, it follows that

$$< d, v >_Y \le [\overline{\mathrm{co}}^* A(y), v]_+ = [A(y), v]_+ .$$

Now let the inequality

$$[A(y), v]_+ \ge < d, v >_Y \quad \forall\, v \in Y$$

be valid and nevertheless $d \notin \overline{\mathrm{co}}^* A(y)$. The set $\overline{\mathrm{co}}^* A(y)$ is convex and closed in $\sigma(Y^*; Y)$-topology of the space Y^*, therefore from the separability theorem there exists $v_0 \in Y$ such that

$$[A(y), v_0]_+ = [\overline{\text{co}}^{*} A(y), v_0]_+ << d, v_0 >_Y$$

which contradicts the condition of the proposition.

Proposition 3. Let $a(\cdot,\cdot): D \subset Y \times Y \to \overline{R} = R \cup \{+\infty\}$. For each $y \in D \subset Y$ a functional $Y \ni w \mapsto a(y, w)$ is positive homogeneous convex and lower semi-continuous if and only if there exists a multi-valued map $A: Y \to 2^{Y^*}$ such that $D(A) = D$ and

$$a(y, w) = [A(y), w]_+ \quad \forall\, y \in D(A),\ w \in Y.$$

Proof. Let $A: D(A) \subset Y \to 2^{Y^*}$. Then for each $y \in D(A)$ the functional $Y \ni v \mapsto a(y, v) = [A(y), v]_+$ is positive homogeneous and semi-additive since the proposition 1. Hence it is convex. Its lower semicontinuity is obvious.

Now let $Y \ni v \mapsto a(y, v)$ be a positive homogeneous convex and lower semicontinuous functional for each $y \in D \subset Y$. Since $a(y, 0) = 0$, it is the point-wise upper bound of a set of continuous linear functionals. We denote this set by $A(y) \subset Y^*$. Thus $a(y, v) = [A(y), v]_+$.

We remind that the multi-valued map $A: D(A) \subset Y \to 2^{Y^*}$ is called monotone if $\forall\, y_1,\ y_2 \in D(A)$ $< d_1 - d_2, y_1 - y_2 >_Y \geq 0$ $\forall\, d_1 \in A(y_1)$, $d_2 \in A(y_2)$.

By using the above mentioned introduced brackets it is easy to note that the multi-valued operator $A: D(A) \subset Y \to 2^{Y^*}$ is monotone if and only if

$$[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_+ \quad \forall\, y_1, y_2 \in D(A).$$

Besides the usual monotony of the multi-valued maps we are interested in:

- N-monotony, i.e.

$$[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_- \quad \forall\, y_1, y_2 \in D(A);$$

- V-monotony, i.e.

$$[A(y_1), y_1 - y_2]_+ \geq [A(y_2), y_1 - y_2]_+ \quad \forall\, y_1, y_2 \in D(A);$$

- w-monotony, i.e.

$$[A(y_1), y_1 - y_2]_+ \geq [A(y_2), y_1 - y_2]_- \quad \forall\, y_1, y_2 \in D(A).$$

Remark 1. Together with the forms a_+, a_- we consider the forms $\overline{a}_+(y, \omega) = [[A(y), \omega]]_+ = \sup_{d \in A(y)} |< d, w >|$ and $\overline{a}_-(y, \omega) = [[A(y), \omega]]_- = \inf_{d \in A(y)} |< d, w >|$ $\forall\, y, \omega \in X$. Thus it is obvious that

$$[A(y), \omega]_+ \leq |[A(y), \omega]_+| \leq [[A(y), \omega]]_+ \leq \| A(y) \|_+ \| \omega \|_X ,$$

$$[A(y), \omega]_- \leq |[A(y), \omega]_-| \leq [[A(y), \omega]]_- \leq \| A(y) \|_- \| \omega \|_X .$$

Remark 2. Further $y_n \xrightarrow{w} y$ in Y will mean that y_n weakly converges to y in the space Y. If Y is not reflexive, then $y_n \xrightarrow{w} y$ in Y^* means that y_n $*$-weakly converges to y in the space Y^*. We denote as $C_v(Y)$ the family of all nonempty closed convex bounded subsets of Y.

Definition 1. Let $D(A)$ be some subset. The multi-valued map $A: D(A) \subset \subset Y \to 2^{Y^*}$ is called:

- $+(-)$-*coercive* if $\| y \|_Y^{-1} [A(y), y]_{+(-)} \to +\infty$ as $\| y \|_Y \to +\infty$, $y \in D(A)$;
- uniformly $+(-)$-coercive if for some $c > 0$

$$\frac{[A(y), y]_{+(-)} - c\| A(y) \|_{+(-)}}{\| y \|_Y} \to +\infty \quad \text{as} \quad \| y \|_Y \to +\infty, \quad y \in D(A);$$

- bounded if for any $L > 0$ there exists $l > 0$ such that $\| A(y) \|_+ \le l$ $\forall\, y \in D(A)$ $\| y \|_Y \le L$;
- locally bounded if for any fixed $y \in D(A)$ there exist the constants $m > 0$ and $M > 0$ such that $\| A(\xi) \|_+ \le M$ when $\| y - \xi \|_Y \le m$, $\xi \in D(A)$;
- d-closed if from the fact that $D(A) \ni y_n \to y \in D(A)$ strongly in Y it follows

$$\underline{\lim}_{n\to\infty} [A(y_n), \varphi]_- \ge [A(y), \varphi]_- \quad \forall\, \varphi \in Y.$$

Let W be also a normalized space with the norm $\| \cdot \|_W$. We consider $W \subset Y$ with continuous embedding.

Definition 2. The multi-valued map $A: D(A) \subset Y \to 2^{Y^*}$ with the convex definitional domain $D(A)$ is called:

- radial lower semicontinuous if for any fixed y, $\xi \in D(A): \xi - y \in D(A)$

$$\underline{\lim}_{t\to+0} [A(y + t\xi), \xi]_+ \ge [A(y), \xi]_-;$$

- radial continuous if the real function $[0, \varepsilon] \ni t \to [A(y + t\xi), \xi]_-$ is continuous from the right in the point $t = 0$ for any fixed y, $\xi \in D(A): \xi - y \in D(A)$;
- radial continuous from above if the real function

$$[0, \varepsilon] \ni t \to [A(y + t\xi), \xi]_+$$

is upper semi-continuous from the right in point $t = 0$ for any fixed y, $\xi \in D(A)$;

- an operator with semi-bounded variation on W (with (Y, W)-semi-bounded variation) if $\forall\, y_1, y_2 \in D(A)$, $\| y_1 \|_Y \le R$, $\| y_2 \|_Y \le R$

$$[A(y_1), y_1 - y_2]_- \ge [A(y_2), y_1 - y_2]_+ - C(R; \| y_1 - y_2 \|'_W;$$

- an operator with N-semi-bounded variation on W if

$$[A(y_1), y_1 - y_2]_- \ge [A(y_2), y_1 - y_2]_- - C(R; \| y_1 - y_2 \|'_W);$$

- an operator with V-semi-bounded variation on W if

$$[A(y_1), y_1 - y_2]_+ \ge [A(y_2), y_1 - y_2]_+ - C(R; \| y_1 - y_2 \|'_W);$$

- λ-pseudomonotone on W (w_λ-pseudomonotone), if for every sequence $\{y_n\}_{n\ge 0} \subset W \cap D(A)$ such that $y_n \xrightarrow{w} y_0$ in W, from the inequality

$$\overline{\lim_{n\to\infty}} < d_n, y_n - y_0 >_Y \le 0, \tag{2.1}$$

where $d_n \in A(y_n)$ $\forall n \ge 1$, it follows the existence of $\{y_{n_k}\}_{k\ge1}$ from $\{y_n\}_{n\ge1}$ and $\{d_{n_k}\}_{k\ge1}$ from $\{d_n\}_{n\ge1}$ such that

$$\underline{\lim_{k\to\infty}} < d_{n_k}, y_{n_k} - w >_Y \ge [A(y_0), y_0 - w]_- \quad \forall w \in D(A); \tag{2.2}$$

- λ_0-pseudomonotone on W (w_{λ_0}-pseudomonotone), if for every sequence $\{y_n\}_{n\ge0} \subset W \cap D(A)$ such that $y_n \xrightarrow{w} y_0$ in W, $d_n \xrightarrow{w} d_0$ in Y^*, where $d_n \in A(y_n)$ $\forall n \ge 1$, from the inequality (2.1), it follows the existence of $\{y_{n_k}\}_{k\ge1}$ from $\{y_n\}_{n\ge1}$ and $\{d_{n_k}\}_{k\ge1}$ from $\{d_n\}_{n\ge1}$ such that (2.2) is true.

The mentioned above multi-valued map satisfies:

- the property $(\kappa)_{+(-)}$, if for every bounded set D in X there exists $c \in R$ such that

$$[A(v), v]_{+(-)} \ge -c \| v \|_X \quad \forall v \in D \setminus \{\bar{0}\}.$$

Here $C \in \Phi$, i.e. $C(r_1; \cdot) : \mathbf{R}_+ \to \mathbf{R}$ is a continuous function for every $r_1 \ge 0$ and such that $\tau^{-1} C(r_1; \tau r_2) \to 0$ for $\tau \to 0+$ $\forall r_1, r_2 \ge 0$ and $\| \cdot \|'_W$ is the (semi) norm on Y, that is relatively compact on W and relatively continuous on Y !

Remark 3. The idea of the passage to a subsequences in the definition 1 was adopted by us from Skripnik's work [15].

Now let $Y = Y_1 \cap Y_2$, where $(Y_1, \| \cdot \|_{Y_1})$ and $(Y_2, \| \cdot \|_{Y_2})$ are Banach spaces.

Definition 3. The pair of the multi-valued maps $A : D(A) \subset Y_1 \to 2^{Y_1^*}$ and $B : D(B) \subset Y_2 \to 2^{Y_2^*}$ is called s *-mutually bounded*, if for every $M > 0$ there exists $K(M) > 0$ such that from

$$\| y \|_Y \le M, \quad y \in D(A) \cap D(B) \text{ and } < d_1(y), y >_{Y_1} + < d_2(y), y >_{Y_2} \le M$$

we have

$$\text{or } \| d_1(y) \|_{Y_1^*} \le K(M), \text{ or } \| d_2(y) \|_{Y_2^*} \le K(M)$$

for some selectors $d_1 \in A$ and $d_2 \in B$.

Remark 4. Further $A : Y \to 2^{Y^*}$ will mean that A maps Y into $2^{Y^*} \setminus \varnothing$, i.e. A is a multi-valued map with nonempty bounded values.

3. THE MAIN PROPERTIES OF THE w_{λ_0}-PSEUDOMONOTONE MAPS

Lemma 1. Let $A : Y_1 \to 2^{Y_1^*}$ and $B : Y_2 \to 2^{Y_2^*}$ be some multi-valued $+(-)$-coercive maps that satisfy the condition $(\kappa)_{+(-)}$. Then the multi-valued operator $C := A + B : Y \to 2^{Y^*}$ is also $+(-)$-coercive.

Proof. We obtain this statement arguing by contradiction. Let $\exists \{y_n\}_{n\ge1} \subset \subset Y$: $\| y_n \|_Y = \| y_n \|_{Y_1} + \| y_n \|_{Y_2} \to +\infty$ as $n \to +\infty$, but

$$\sup_{n\ge1} \frac{[C(y_n), y_n]_{+(-)}}{\| y_n \|_Y} < +\infty . \tag{3.1}$$

Case 1. $\| y_n \|_{Y_1} \to +\infty$ as $n \to \infty$, $\| y_n \|_{Y_2} \le c \quad \forall n \ge 1$.

$$\gamma_A(r) := \inf_{\|v\|_{Y_1} = r} \frac{[A(v), v]_{+(-)}}{\| v \|_{Y_1}}, \quad \gamma_B(r) := \inf_{\|w\|_{Y_2} = r} \frac{[B(w), w]_{+(-)}}{\| w \|_{Y_2}}, \quad r > 0.$$

We remark that $\gamma_A(r) \to +\infty,\ \gamma_B(r) \to +\infty$ as $r \to +\infty$. Then $\forall n \ge 1$

$$\| y_n \|_{Y_1}^{-1} [A(y_n), y_n]_{+(-)} \ge \gamma_A(\| y_n \|_{Y_1}) \| y_n \|_{Y_1} \text{ and } \frac{[A(y_n), y_n]_{+(-)}}{\| y_n \|_Y} \ge$$

$$\ge \gamma_A(\| y_n \|_{Y_1}) \frac{\| y_n \|_{Y_1}}{\| y_n \|_Y} \to +\infty \text{ as } \| y_n \|_{Y_1} \to +\infty \text{ and } \| y_n \|_{Y_2} \le c .$$

Due to the condition $(\kappa)_{+(-)}$ for every $n \ge 1$

$$\frac{[B(y_n), y_n]_{+(-)}}{\| y_n \|_Y} \ge \gamma_B(\| y_n \|_{Y_2}) \frac{\| y_n \|_{Y_2}}{\| y_n \|_Y} \ge -c_1 \frac{\| y_n \|_{Y_2}}{\| y_n \|_Y} \to 0 \text{ as } n \to \infty,$$

where $c_1 \in R$ is a constant as in the condition $(\kappa)_{+(-)}$ with

$$D = \left\{ y \in Y_2 \,\middle|\, \| y \|_{Y_2} \le c \right\}.$$

It is obvious that

$$\frac{[C(y_n), y_n]_{+(-)}}{\| y_n \|_Y} = \frac{[A(y_n), y_n]_{+(-)}}{\| y_n \|_Y} + \frac{[B(y_n), y_n]_{+(-)}}{\| y_n \|_Y} \to +\infty \quad \text{as} \quad n \to \infty.$$

This is in contradiction with (3.1).

Case 2. The case when $\| y_n \|_{Y_1} \le c \quad \forall n \ge 1$ and $\| y_n \|_{Y_2} \to \infty$ as $n \to +\infty$ can be examined in the same way.

Case 3. Let us consider the situation $\| y_n \|_{Y_1} \to +\infty$ and $\| y_n \|_{Y_2} \to +\infty$ as $n \to +\infty$. Then

$$+\infty > \sup_{n\ge1} \frac{[C(y_n), y_n]_{+(-)}}{\| y_n \|_Y} \ge \gamma_A(\| y_n \|_{Y_1}) \frac{\| y_n \|_{Y_1}}{\| y_n \|_{Y_1} + \| y_n \|_{Y_2}} +$$

$$+ \gamma_B(\| y_n \|_{Y_2}) \frac{\| y_n \|_{Y_2}}{\| y_n \|_{Y_1} + \| y_n \|_{Y_2}} . \tag{3.2}$$

It is obvious that $\forall\, n \ge 1$ $\dfrac{\| y_n \|_{Y_1}}{\| y_n \|_Y} > 0$ and $\dfrac{\| y_n \|_{Y_2}}{\| y_n \|_Y} > 0$ and moreover if even one of the boundaries, for example, $\dfrac{\| y_n \|_{Y_1}}{\| y_n \|_Y} \to 0$, then $\dfrac{\| y_n \|_{Y_2}}{\| y_n \|_Y} = 1 - \dfrac{\| y_n \|_{Y_1}}{\| y_n \|_Y} \to 1$. Then we have a contradiction in (3.2).

Lemma 2. Every strict multi-valued operator $A : Y \to 2^{Y^*}$ with $(Y;W)$-semi-bounded variation is bounded-valued, i.e. $A : Y \to 2^{Y^*}$.

Proof. We remark that for every $y \in Y$

$$Y \ni \omega \to [A(y), \omega]_+ \in \mathbf{R} \cup \{+\infty\}, \quad Y \ni \omega \to [A(y), \omega]_- \in \mathbf{R} \cup \{-\infty\}.$$

So, due to the definition of the semi-bounded variation on (Y, W) we obtain that for all $\omega \in Y$, for some $R = R(\omega, y) > 0$

$$[A(y), \omega]_+ \le [A(y + \omega), \omega]_- + C_A(R; \| \omega \|'_W) < +\infty.$$

From last, in virtue of Banach-Steinhaus theorem, it follows that $\| A(y) \|_+ < < +\infty$ for every $y \in Y$.

Lemma 3. The multi-valued operator $A : Y \to 2^{Y^*}$ with $(Y;W)$-semi-bounded variation is locally bounded.

Proof. We obtain this statement arguing by contradiction. If A is not locally bounded then for some $y \in Y$ there exists a sequence $\{y_n\}_{n \ge 1} \subset Y$ such that $y_n \to y$ in Y and $\| A(y_n) \|_+ \to +\infty$ as $n \to +\infty$. We suppose that

$$\alpha_n = 1 + \| A(y_n) \|_+ \| y_n - y \|_Y$$

for every $n \ge 1$. Then, due to the proposition 1, $\forall\, \omega \in Y$ and some $R > 0$ we have

$$\alpha_n^{-1} [A(y_n), \omega]_+ \le \alpha_n^{-1} \{ [A(y_n), y_n - y]_+ + [A(y_n), \omega + y - y_n]_+ \} \le$$

$$\le \alpha_n^{-1} \{ [A(y_n), y_n - y]_+ + [A(y + \omega), y + \omega - y_n]_+ + C_A(R; \| y_n - y - \omega \|'_W) \}.$$

Since the sequence $\{\alpha_n^{-1}\}$ is bounded and $\| y_n - y - \omega \|'_W \to \| \omega \|'_W$ (according to the assumption $\| \xi \|'_W \le k \| \xi \|_Y$ for all $y \in Y$), due to proposition 1, we have

$$\forall\, n \ge 1 \quad \alpha_n^{-1} [A(y_n), \omega]_+ \le \alpha_n^{-1} \{ C_A(R; \| y_n - y - \omega \|'_W) +$$

$$+ \| A(y + \omega) \|_+ \cdot \| y + \omega - y_n \|_Y \} + 1 \le N_1,$$

where N_1 does not depend on $n \ge 1$. Thus,

$$\sup_{n \ge 1} \left| \alpha_n^{-1} [A(y_n), \omega]_+ \right| < \infty \quad \forall\, \omega \in Y.$$

Therefore, since the Banach-Steinhaus theorem, there exists $N > 0$ such that

$$\| A(y_n) \|_+ \le N \alpha_n = N \left(1 + \| A(y_n) \|_+ \cdot \| y_n - y \|_Y \right) \quad \forall\, n \ge 1.$$

By choosing $n_0 \geq 1$ from the condition $N \| y_n - y \| \leq 1/2 \quad \forall n \geq n_0$ we obtain that for every $n \geq n_0$ $\| A(y_n) \|_+ \leq 2N$, which contradicts the assumption. So, the local boundedness is proved.

Lemma 4. The multi-valued operator $A : Y \to 2^{Y^*}$ with $(Y;W)$-semi-bounded variation has the property (Π): if for some $k_1, k_2 > 0$ and $d \in A$

$$< d(y), y >_Y \leq k_1 \quad \text{for each} \quad y \in Y : \| y \|_Y \leq k_2$$

then there exists $C > 0$ such that

$$\| d(y) \|_{Y^*} \leq C \quad \text{for all} \quad y \in Y : \| y \|_Y \leq k_2 .$$

Proof. In virtue of the locally boundedness of A there exist $\varepsilon > 0$ and $M_\varepsilon > 0$ such that $\| A(\xi) \|_+ \leq M_\varepsilon \quad \forall \| \xi \|_Y \leq \varepsilon$. It means that for some $R \geq \varepsilon$

$$\| d(y) \|_{Y^*} = \sup_{\|\xi\|_Y \leq \varepsilon} \frac{1}{\varepsilon} < d(y), \xi >_Y \leq \sup_{\|\xi\|_Y \leq \varepsilon} \frac{1}{\varepsilon} \left\{ [A(y), \xi - y]_+ + < d(y), y >_Y \right\} \leq$$

$$\leq \sup_{\|\xi\|_Y \leq \varepsilon} \frac{1}{\varepsilon} \{ [A(\xi), \xi - y]_- + < d(y), y >_Y + C_A(R; \| y - \xi \|'_W) \} \leq$$

$$\leq \sup_{\|\xi\|_Y \leq \varepsilon} \frac{1}{\varepsilon} \{ \| A(\xi) \|_+ \cdot \| \xi - y \|_Y + < d(y), y >_Y + C_A(R; \| y - \xi \|'_W) \} \leq$$

$$\leq \frac{1}{\varepsilon} (\varepsilon M_\varepsilon + k_2 M_\varepsilon + k_1 + l) = C ,$$

where $l = \sup_{\|y\|_Y \leq k_2} \sup_{\|\xi\|_Y \leq \varepsilon} C_A(R; \| y - \xi \|'_W) < +\infty$, since $C(R; \cdot) : \mathbf{R}_+ \to \mathbf{R}$ is a continuous function and $\| \cdot \|'_W$ is relatively continuous $\| \cdot \|_Y$ on Y.

Remark 5. It is obvious that if one of the maps of the pair $A, B : X \rightrightarrows X^*$ is bounded, then the pair $(A;B)$ is s-mutually bounded. Moreover, if the pair $(A;B)$ is s-mutually bounded and each of them satisfies the condition (Π), then the operator $C = A + B : X \rightrightarrows X^*$ satisfies the property (Π).

Lemma 5. Let Y be a reflexive Banach space. Then every λ-pseudo-monotone on W map is λ_0-pseudomonotone on W. For bounded maps the converse implication is true.

Proof. The direct implication is obvious. Let us prove the converse implication. We consider the λ_0-pseudomonotone on W map $A : Y \rightrightarrows Y^*$, $y_n \to y$ weakly in W, the (2.1) holds, where $d_n \in \overset{*}{\overline{\text{co}}} A(y_n)$. From the boundedness of the operator A it immediately follows the boundedness of $\overset{*}{\overline{\text{co}}} A$ and so the boundedness of the sequence $\{d_n\}$ in Y^*. Consequently, there exists a subsequence $\{d_m\} \subset \{d_n\}$ and, respectively, $\{y_m\} \subset \{y_n\}$, such that $d_m \to d$ weakly in Y^* and at the same time

$$\overline{\lim_{m\to\infty}} < d_m, y_m - v >_Y \le \overline{\lim_{n\to\infty}} < d_n, y_n - v >_Y \le 0.$$

However the operator A is λ_0-pseudomonotone on W, therefore there exist the subsequences $\{y_{n_k}\}_{k\ge1} \subset \{y_m\}$ and $\{d_{n_k}\}_{k\ge1} \subset \{d_m\}$ for what (2.2) is true. This proves our statement.

Remark 6. Let us pay our attention on the fact that for the classical definitions (not passing to the subsequences) this statement is problematically!

In F. Browder and P. Hess work [16] the class of generous pseudomonotone operators has been introduced.

Definition 4. The operator $A: Y \to C_v(Y^*)$ is called generous pseudomonotone on W, if for each pair of sequences $\{y_n\}_{n\ge1} \subset W$ and $\{d_n\}_{n\ge1} \subset Y^*$ such that $d_n \in A(y_n)$, $y_n \to y$ weakly in W, $d_n \to d$ $*$-weakly in Y^*, from the inequality

$$\overline{\lim_{n\to\infty}} < d_n, y_n >_Y \le < d, y >_Y \tag{3.3}$$

we have $d \in A(y)$ and $< d_n, y_n >_Y \to < d, y >_Y$.

Proposition 4. Every generous pseudomonotone on W operator is λ_0-pseudomonotone on W.

Proof. Let $y_n \to y$ weakly in W, $A(y_n) \ni d_n \to d$ $*$-weakly in Y^* and (3.3) holds (we remark that in this case the inequality (2.1) is also true). Then, in view of the generous pseudomonotony, $< d_n, y_n >_Y \to < d, y >_Y$, $d \in A(y)$, consequently, in virtue of the proposition 2,

$$\underline{\lim_{n\to\infty}} < d_n, y_n - v >_Y = < d, y - v >_Y \ge [A(y), y - v]_- \quad \forall v \in Y.$$

The converse statement in the proposition is not true, but

Proposition 5. Let $A: Y \to 2^{Y^*}$ be a λ_0-pseudomonotone operator. Then the next property takes place: from $y_n \to y$ weakly in W, $\overline{\text{co}}^* A(y_n) \ni d_n \to d$ $*$-weakly in Y^* and from the inequality (2.1) the existence of the subsequences $\{y_m\} \subset \{y_n\}$ and $\{d_m\} \subset \{d_n\}$ such that $< d_m, y_m >_Y \to < d, y >_Y$, with $d \in \overline{\text{co}}^* A(y)$, follows.

Proof. Let $\{y_n\}, \{d_n\}$ be required sequences, consequently, one can choose such subsequences $\{y_m\}$, $\{d_m\}$, that the inequality (2.2) is true. By fixing in the last relation $\omega = y$, we get

$$< d_m, y_m - y >_Y \to 0 \text{ or } < d_m, y_m >_Y \to < d, y >_Y,$$

$$< d, y - v >_Y = \lim_{m\to\infty} < d_m, y_m - v >_Y \ge [A(y), y - v]_- \ \forall\, v \in Y.$$

From here, in virtue of the proposition 2 we obtain that $d \in \overline{\text{co}}^* A(y)$.

Proposition 6. Let $A = A_0 + A_1 : Y \rightrightarrows Y^*$, where $A_0 : Y \rightrightarrows Y^*$ is a monotone map, and the operator $A_1 : Y \rightrightarrows Y^*$ has the following properties:

1) there exists a linear normalized space Z in which W is compactly and densely enclosed and $Y \subset Z$ with continuous and dense embedding;

2) the operator $A_1 : Z \rightrightarrows Z^*$ univocal and locally polynomial, i.e. $\forall\, R > 0$ there exists $n = n(R)$ and a polynomial function $P_R(t) = \sum\limits_{0<\alpha\le n} \lambda_\alpha(R) t^\alpha$ with continuous factors $\lambda_\alpha(R) \ge 0$ such that the estimation is valid

$$\|A_1(y_1) - A_1(y_2)\|_+^{(Z^*)} \le P_R\left(\|y_1 - y_2\|_Z\right) \ \forall\ \|y_i\|_Z \le R,\ i = 1,2.$$

Then A is the operator with semi-bounded variation on W.

Proposition 7. Let in the previous proposition the operator $A_0 : Y \rightrightarrows Y^*$ be N-monotone, and instead of the condition 2) we make the following one:

2') a map (multi-valued) $A_1 : Z \rightrightarrows Z^*$ is locally polynomial in the sense that $\forall\, R > 0$ there exists $n = n(R)$ and a polynomial $P_R(t)$ for which

$$\mathrm{dist}\left(A_1(y_1), A_1(y_2)\right) \le P_R\left(\|y_1 - y_2\|_Z\right) \ \forall \| y_i||_Z \le R,\ i = 1,2. \qquad (3.4)$$

Then $A = A_0 + A_1$ is the operator with N-semi-bounded variation on W.

Proof. We give the proof in the proposition 7. In the case of the proposition 6 the reasonings are similar. Since for each $y_1, y_2 \in Y$

$$[A_0(y_1), y_1 - y_2]_- \ge [A_0(y_2), y_1 - y_2]_+,$$

we must estimate $[A_1(y_1), y_1 - y_2]_- - [A_1(y_2), y_1 - y_2]_-$.

For any $d_1 \in A_1(y_1)$, $d_2 \in A_1(y_2)$ we find

$$< d_2, y_1 - y_2 >_Y - < d_1, y_1 - y_2 >_Y = < d_2, y_1 - y_2 >_Z - < d_1, y_1 - y_2 >_Z \le$$
$$\le \|d_1 - d_2\|_{Z^*} \|y_1 - y_2\|_Z,$$

hence

$$[A_1(y_2), y_1 - y_2]_- - [A_1(y_1), y_1 - y_2]_- \le \mathrm{dist}\left(A_1(y_1), A_1(y_2)\right)\|y_1 - y_2\|_Z.$$

From here and from the estimation (3.4) as $\|y_i\|_Z \le R$ $(i = 1,2)$ (respectively $\|y_i\|_Y \le \hat{R}$, $R = R(\hat{R})$) we obtain

$$[A_1(y_1), y_1 - y_2]_- \ge [A_1(y_2), y_1 - y_2]_- - C\left(\hat{R}; \| y_1 - y_2||'_W\right),$$

where $\|\cdot\|'_W = \|\cdot\|_Z$, $C(R,t) = P_R(t)t$.

It is easy to check that $C \in \Phi$.

Proposition 8. Let one of two conditions hold:

1) $A : Y \rightrightarrows Y^*$ is radially lower semi-continuous operator with semi-bounded variation on W;

2) $A : Y \rightrightarrows Y^*$ is radially continuous from above operator with N-semi-bounded variation on W with compact values in Y^*.

Then A is λ_0-pseudomonotone on W map.

Proof. Let $y_n \to y$ weakly in W, $\overset{*}{\overline{\mathrm{co}}} A(y_n) \ni d_n \to d$ $*$-weakly in Y^* and (2.1) is true. By using the property of semi-bounded variation on W of the operator A, we conclude that for every $v \in Y$

$$< d_n, y_n - v >_Y \geq [A(y_n), y_n - v]_- \geq [A(v), y_n - v]_+ - C\left(R; \|y_n - v\|'_W\right).$$

The function $X \ni w \mapsto [A(v), w]_+$ is convex and semi-continuous from below, and so it is weakly semi-continuous from below, therefore by substituting in the last inequality $v = y$ and passing to the limit, in view of the properties of the function C, we obtain $\varliminf\limits_{n\to\infty} < d_n, y_n - y >_Y \geq 0$, i.e $< d_n, y_n - y >_Y \to 0$.

For any $h \in Y$ and $\tau \in [0,1]$ we shall put $\omega_\tau = \tau h + (1-\tau) y$, then

$$< d_n, y_n - \omega_\tau >_Y \geq [A(\omega_\tau), y_n - \omega_\tau]_+ - C\left(R; \|y_n - \omega_\tau\|'_W\right)$$

or by passing to the limit

$$\tau \varliminf_{n\to\infty} < d_n, y - h >_Y \geq \tau [A(w_\tau), y - h]_+ - C\left(R; \tau \|y - h\|'_W\right).$$

By dividing the last inequality by τ and by passing to the limit as $\tau \to 0+$, in view of the radial lower semi-continuity of A and of the properties of the function C, we obtain that for each $h \in Y$

$$\varliminf_{n\to\infty} < d_n, y - h >_Y \geq$$

$$\geq \varliminf_{\tau\to+0} [A(\omega_\tau), y - h]_+ + \lim_{\tau\to+0} \frac{1}{\tau} C\left(R; \tau \|y - h\|'_W\right) \geq [A(y), y - h]_-.$$

Moreover as $< d_n, y_n - y >_Y \to 0$ we get

$$\varliminf_{n\to\infty} < d_n, y_n - h >_Y = \varliminf_{n\to\infty} < d_n, y - h >_Y \geq [A(y), y - h]_- \quad \forall h \in Y,$$

and this proves the first statement of the proposition 8.

Now we stop on the basic distinctive moments of the second statement. Because of the N-semi-boundedness of the variation for the operator A we conclude that

$$\varliminf_{n\to\infty} < d_n, y_n - v >_Y \geq \varliminf_{n\to\infty} [A(y_n), y_n - v]_- \geq$$

$$\geq \varliminf_{n\to\infty} [A(v), y_n - v]_- - C\left(R; \|y - v\|'_W\right). \tag{3.5}$$

Let us estimate the first member in the right part of (3.5). Let us prove that the function $X \ni h \mapsto [A(v), h]_-$ is weakly lower semi-continuous $\forall v \in Y$. Let $z_n \to z$ weakly in Y, then for each $n = 1,2,\ldots$ $\exists \xi_n \in \overset{*}{\overline{\mathrm{co}}} A(v)$ such that

$$[A(v), z_n]_- = < \xi_n, z_n >_Y.$$

From the sequence $\{\xi_n; z_n\}$ we take a subsequence $\{\xi_m; z_m\}$ such that

$$\varliminf_{n\to\infty} [A(v), z_n]_- = \varliminf_{n\to\infty} < \xi_n, z_n >_Y = \lim_{m\to\infty} < \xi_m, z_m >_Y$$

and by virtue of the compactness of the set $\overline{\text{co}}^* A(v)$ we find that $\xi_m \to \xi$ strongly in Y^* with $\xi \in \overline{\text{co}}^* A(v)$. Hence

$$\varliminf_{n\to\infty} [A(v), z_n]_- = \lim_{n\to\infty} < \xi_m, z_m >_Y = < \xi, z >_Y = [A(v), z]_-,$$

and this proves the weak lower semi-continuity of the function $h \mapsto [A(v), h]_-$.

So from (3.5) we get

$$\varliminf_{n\to\infty} < d_n, y_n - v >_Y \geq \varliminf_{n\to\infty} [A(y_n), y_n - v]_- \geq [A(v), y - v]_- - C\left(R; \|y - v\|_W{}'\right).$$

Then by substituting v with y in the last inequality we have $< d_n, y_n - y >_X \to 0$, therefore

$$\varliminf_{n\to\infty} < d_n, y_n - v >_Y \geq [A(v), v - w]_- - C\left(R; \|y - v\|_Y^{'}\right) \quad \forall v \in Y.$$

By substituting in the last inequality v with $tw + (1-t)y$, where $w \in Y$, $t \in [0,1]$, then by dividing the result on t and by passing to the limit as $t \to +0$, because of the radial semi-continuity from above we find

$$\varliminf_{n\to\infty} < d_n, y_n - w >_Y \geq [A(y), y - w]_- \quad \forall w \in Y.$$

Now let $W = W_1 \cap W_2$, where $(W_1, \|\cdot\|_{W_1})$ and $(W_2, \|\cdot\|_{W_2})$ are Banach spaces such that $W_i \subset Y_i$ with continuous embedding.

Lemma 6. Let Y_1, Y_2 be reflexive Banach spaces, $A: Y_1 \to C_v(Y_1^*)$ and $B: Y_2 \to C_v(Y_2^*)$ be s-mutually bounded λ_0-pseudomonotone respectively on W_1 and on W_2 multivalued maps. Then $C := A + B: Y \to C_v(Y^*)$ is λ_0-pseudomonotone on W map.

Remark 7. If the pair ($A; B$) is not s-mutually bounded, then the last proposition holds only for λ-pseudomonotone (respectively on W_1 and on W_2) maps.

Proof. At first we check that $\forall y \in Y \quad C(y) \in C_v(Y^*)$. The convexity of $C(y)$ follows from the same property for $A(y)$ and $B(y)$. By virtue of the Mazur theorem, it is enough to prove that the set $C(y)$ is weakly closed. Let c be a frontier point of $C(y)$ with respect to the topology $\sigma(Y^*; Y^{**}) = \sigma(Y^*; Y)$ (the space Y is reflexive). Then

$$\exists \{c_m\}_{m\geq 1} \subset C(y): \quad c_m \to c \quad \text{weakly in} \quad Y^* \quad \text{as} \quad m \to +\infty.$$

From here, since the maps A and B have bounded values, due to the Banach-Alaoglu theorem, we can assume that for each $m \geq 1$ there exist $v_m \in A(y)$ and $w_m \in B(y)$ such that $v_m + w_m = c_m$ and by passing (if it is necessary) to the subsequences we obtain:

$$v_m \xrightarrow{w} v \quad in \quad Y_1^* \quad \text{and} \quad w_m \xrightarrow{w} w \quad \text{in} \quad Y_2^*$$

for some $v \in A(y)$ and $w \in B(y)$. Hence $c = v + w \in C(y)$. So it is proved that the set $C(y)$ is weakly closed in Y^*.

Now let $y_n \xrightarrow{w} y_0$ in W (from here it follows that $y_n \xrightarrow{w} y_0$ in W_1 and $y_n \xrightarrow{w} y_0$ in W_2), $C(y_n) \ni d(y_n) \xrightarrow{w} d_0$ in Y^* and the inequality (2.1) be true. Hence

$$d_A(y_n) \in A(y_n) \quad \text{and} \quad d_B(y_n) \in B(y_n): \quad d_A(y_n) + d_B(y_n) = d(y_n).$$

Since the pair ($A; B$) is s -mutually bounded, from the estimation

$$< d(y_n), y_n >_Y =< d_A(y_n) + d_B(y_n), y_n >_Y =$$

$$=< d_A(y_n), y_n >_{Y_1} + < d_B(y_n), y_n >_{Y_2} \leq k$$

we have or $\| d_A(y_n) \|_{Y_1^*} \leq C$ or $\| d_B(y_n) \|_{Y_2^*} \leq C$. Then, due to the reflexivity of Y_1 and Y_2, by passing (if it is necessary) to a subsequence we get

$$d_A(y_n) \xrightarrow{w} d_0' \quad \text{in} \quad Y_1^* \quad \text{and} \quad d_B(y_n) \xrightarrow{w} d_0'' \quad \text{in} \quad Y_2^*. \qquad (3.6)$$

From the inequality (2.1) we have

$$\underline{\lim}_{n\to\infty} < d_B(y_n), y_n - y_0 >_{Y_2} + \overline{\lim}_{n\to\infty} < d_A(y_n), y_n - y_0 >_{Y_1} \leq$$

$$\leq \overline{\lim}_{n\to\infty} < d(y_n), y_n - y_0 >_Y \leq 0,$$

or symmetrically

$$\underline{\lim}_{n\to\infty} < d_A(y_n), y_n - y_0 >_{Y_1} + \overline{\lim}_{n\to\infty} < d_B(y_n), y_n - y_0 >_{Y_2} \leq$$

$$\leq \overline{\lim}_{n\to\infty} < d(y_n), y_n - y_0 >_Y \leq 0.$$

Let us consider the last inequality. It is obvious that there exists a subsequence $\{y_m\}_m \subset \{y_n\}_{n\geq 1}$ such that

$$0 \geq \overline{\lim}_{n\to\infty} < d_B(y_n), y_n - y_0 >_{Y_2} + \underline{\lim}_{n\to\infty} < d_A(y_n), y_n - y_0 >_{Y_1} \geq$$

$$\geq \overline{\lim}_{m\to\infty} < d_B(y_m), y_m - y_0 >_{Y_2} + \lim_{m\to\infty} < d_A(y_m), y_m - y_0 >_{Y_1}. \qquad (3.7)$$

From here we obtain:

$$\text{or} \quad \lim_{m\to\infty} < d_A(y_m), y_m - y_0 >_{Y_1} \le 0, \quad \text{or} \quad \overline{\lim_{m\to\infty}} < d_B(y_m), y_m - y_0 >_{Y_2} \le 0 .$$

Without loss of generality we suppose that

$$\lim_{m\to\infty} < d_A(y_m), y_m - y_0 >_{Y_1} \le 0 .$$

Then because of (3.6) and of the λ_0-pseudomonotony of A on W_1 there exists a subsequence $\{y_{m_k}\}_{k\ge1} \subset \{y_m\}_m$ such that

$$\lim_{k\to\infty} < d_A(y_{m_k}), y_{m_k} - v >_{Y_1} \ge [A(y_0), y_0 - v]_- \quad \forall\, v \in Y_1 . \tag{3.8}$$

By substituting in the last relation v with y_0 it results in

$$< d_A(y_{m_k}), y_{m_k} - y_0 >_{Y_1} \to 0 \quad \text{as} \quad k \to +\infty .$$

Therefore, taking into account (3.7), we have

$$\overline{\lim_{k\to\infty}} < d_B(y_{m_k}),\ y_{m_k} - y_0 >_{Y_2} \le 0.$$

By virtue of the λ_0-pseudomonotony of B on W_2, by passing to a subsequence $\{y_{m_{k'}}\} \subset \{y_{m_k}\}_{k\ge1}$ we find

$$\underline{\lim_{k\to\infty}} < d_B(y_{m_{k'}}), y_{m_{k'}} - w >_{Y_2} \ge [B(y_0), y_0 - w]_- \quad \forall\, w \in Y_2 . \tag{3.9}$$

So from the relations (3.8) and (3.9) we finally obtain

$$\underline{\lim_{k\to\infty}} < d(y_{m_{k'}}), y_{m_{k'}} - x >_Y \ge \lim_{k\to\infty} < d_A(y_{m_{k'}}), y_{m_{k'}} - x >_{Y_1} +$$

$$+ \underline{\lim_{k\to\infty}} < d_B(y_{m_{k'}}), y_{m_{k'}} - x >_{Y_2} \ge [A(y_0), y_0 - x]_- +$$

$$+ [B(y_0), y_0 - x]_- = [C(y_0), y_0 - x]_- \quad \forall x \in Y .$$

Proposition 9. Every $-$-coercive multi-valued map $A : Y \to 2^{Y^*}$ is $+$-coercive; every monotone $+$-coercive multi-valued map is $-$-coercive, uniformly $-$-coercive and uniformly $+$-coercive.

Proof. The first part of this proposition is the direct corollary of the definitions of $[\cdot,\cdot]_+$ and of $[\cdot,\cdot]_-$. Let us check the second one.

Let $A : Y \to 2^{Y^*}$ be a monotone $+(-)$-coercive map. Let us prove that A is uniformly $+(-)$-coercive.

From the lemma 3 it follows that there exist a ball

$$\overline{B}_r = \{y \in Y \mid \| y \|_Y \le r\}$$

and a constant $c_1 > 0$ such that

$$\| A(\omega) \|_+ \le c_1 \quad \forall\, \omega \in \overline{B}_r .$$

Hence for each $y \in Y$

$$\| A(y) \|_+ = \sup_{d(y)\in A(y)} \sup_{\omega\in\overline{B}_r} \frac{1}{r} < d(y), \omega >_Y = \frac{1}{r} \sup_{\omega\in\overline{B}_r} [A(y), \omega]_+ \le$$

$$\le \frac{1}{r} \sup_{\omega\in\overline{B}_r} \{[A(y), \omega - y]_+ + [A(y), y]_+\} \le \frac{1}{r} \sup_{\omega\in\overline{B}_r} \{[A(\omega), \omega - y]_+ +$$

$$+ [A(y), y]_+\} \le \frac{1}{r}\{c_1(r + \| y \|_Y) + [A(y), y]_+\} =$$

$$= \frac{1}{r}[A(y), y]_+ + c_1 + \frac{c_1}{r} \| y \|_Y ;$$

$$\| A(y) \|_- = \inf_{d(y)\in A(y)} \sup_{\omega\in\overline{B}_r} \frac{1}{r} < d(y), \omega >_Y =$$

$$= \inf_{d(y)\in A(y)} \sup_{\omega\in\overline{B}_r} \frac{1}{r}\{< d(y), \omega - y >_Y + < d(y), y >_Y\} \le$$

$$\le \inf_{d(y)\in A(y)} \sup_{\omega\in\overline{B}_r} \frac{1}{r}\{< d(\omega), \omega - y >_Y + < d(y), y >_Y\} \le$$

$$\le \frac{1}{r} \inf_{d(y)\in A(y)} \{< d(\omega), y >_Y + c_1(r + \| y \|_X)\} = \frac{1}{r}[A(y), y]_- + c_1 + \frac{c_1}{r} \| y \|_Y ,$$

i.e.

$$\forall y \in Y \ \| A(y) \|_{+(-)} \le \frac{1}{r}[A(y), y]_{+(-)} + c_1 + \frac{c_1}{r} \| y \|_Y .$$

Thus as $c = \frac{r}{2} > 0$ the uniform $+(-)$-coercivity for A follows from the following estimations:

$$\frac{[A(y), y]_{+(-)} - c \| A(y) \|_{+(-)}}{\| y \|_Y} \ge$$

$$\ge \frac{[A(y), y]_{+(-)} - \frac{1}{2}[A(y), y]_{+(-)} - \frac{rc_1}{2} - \frac{c_1}{2} \| y \|_Y}{\| y \|_Y} =$$

$$= \frac{\left[A(y), y - \frac{1}{2} y\right]_- - \frac{rc_1}{2} - \frac{c_1}{2} \| y \|_Y}{\| y \|_Y} \ge$$

$$\ge \frac{\left[A\left(\frac{1}{2} y\right), y - \frac{1}{2} y\right]_+ - \frac{rc_1}{2} - \frac{c_1}{2} \| y \|_Y}{\| y \|_Y} =$$

$$=\frac{\left[A\left(\frac{1}{2}y\right),y-\frac{1}{2}y\right]_+}{2\|\frac{1}{2}y\|_Y}-\frac{c_1}{2}-\frac{rc_1}{2\|y\|_Y}\to+\infty\quad\text{as}\quad\|y\|_Y\to\infty.$$

To finish the proof it is enough to show that every monotone $+$-coercive map is $-$-coercive. This follows from the next estimations:

$$\frac{[A(y),y]_-}{\|y\|_Y}=\frac{2[A(y),y-\frac{1}{2}y]_-}{\|y\|_Y}\geq\frac{2[A(\frac{1}{2}y),y-\frac{1}{2}y]_+}{\|y\|_Y}=$$

$$=\frac{\left[A\left(\frac{1}{2}y\right),\frac{1}{2}y\right]_+}{\|\frac{1}{2}y\|_Y}\to+\infty\quad\text{as}\quad\|y\|_Y\to+\infty.$$

Corollary 1. Let $\varphi:Y\to R$ be a convex lower semicontinuous functional such that

$$\frac{\varphi(y)}{\|y\|_Y}\to+\infty\quad\text{as}\quad\|y\|_Y\to\infty.$$

Then its subdifferential map

$$\partial\varphi(y)=\{p\in Y^*|<p,\omega-y>_Y\leq\varphi(\omega)-\varphi(y)\quad\forall\,\omega\in Y\}\neq\varnothing,\quad y\in Y$$

is $+$-coercive, and hence, $-$-coercive, uniformly $-$-coercive and uniformly $+$-coercive.

Proof. Due to the monotony of the map $\partial\varphi:Y\to 2^{Y^*}$ and to the proposition 9, it is enough to prove only that it is +-coercive. This follows from the next estimations:

$$\|y\|_Y^{-1}[\partial\varphi(y),y]_+\geq\|y\|_Y^{-1}\varphi(y)-\|y\|_Y^{-1}\varphi(0)\to+\infty\ \text{as}\ \|y\|_Y\to+\infty\,.$$

Definition 5. The multi-valued map $A:Y\to 2^{Y^*}$ satisfies the uniform property $(\kappa)_{+(-)}$ if for each bounded set D in Y and for each $c>0$ there exists $c_1>0$ such that

$$\|A(v)\|_{+(-)}\leq\frac{1}{c}[A(v),v]_{+(-)}+\frac{c_1}{c}\|v\|_Y\quad\forall\,v\in D\setminus\{\bar{0}\}\,.$$

Lemma 7. Let $A:Y_1\rightrightarrows Y_1^*$, $B:Y_2\rightrightarrows Y_2^*$ be +-coercive maps, which satisfy the uniform property $(\kappa)_{+(-)}$. Then the map $C:=A+B:Y\to 2^{Y^*}$ is uniformly $+(-)$-coercive.

Proof. We obtain this statement arguing by contradiction. Let $\{x_n\}_{n\geq1}\subset Y$ with $x_n\neq\bar{0}$ and $\|x_n\|_Y=\|x_n\|_{Y_1}+\|x_n\|_{Y_2}\to+\infty$ as $n\to+\infty$. Taking into account that

$$\sup_{n\geq 1}\frac{[C(x_n),x_n]_{+(-)}-c_C\,\|C(x_n)\|_{+(-)}}{\|x_n\|_Y}<+\infty\,, \qquad (3.10)$$

where $c_C=\min\{c_A,c_B\}$, $c_A,c_B>0$ are the constants as in the uniform +(-)-coercive condition for A and B respectively. Let

$$\gamma_A(r):=\inf_{\|v\|_{Y_1}=r}\frac{[A(v),v]_{+(-)}-c_A\,\|A(v)\|_{+(-)}}{\|v\|_{Y_1}}\,,$$

$$\gamma_B(r):=\inf_{\|w\|_{Y_2}=r}\frac{[B(w),w]_{+(-)}-c_B\,\|B(w)\|_{+(-)}}{\|w\|_{Y_2}}\,,\quad r>0\,,$$

we remark that $\gamma_A(r)\to+\infty,\ \gamma_B(r)\to+\infty$ as $r\to+\infty$. In the case $\|x_n\|_{Y_1}\to+\infty$ as $n\to+\infty$ and $\|x_n\|_{Y_2}\leq c\ \ \forall n\geq 1$ we get

$$\frac{[A(x_n),x_n]_{+(-)}-c_A\,\|A(x_n)\|_{+(-)}}{\|x_n\|_Y}\geq\gamma_A(\|x_n\|_{Y_1})\frac{\|x_n\|_{Y_1}}{\|x_n\|_Y}\to+\infty\quad\text{as}\quad n\to+\infty$$

and moreover

$$\frac{[B(x_n),x_n]_{+(-)}-c_B\,\|B(w)\|_{+(-)}}{\|x_n\|_Y}\geq -c_1\frac{\|x_n\|_{Y_2}}{\|x_n\|_Y}\to 0\ \text{ as }\ +n\to\infty\,,$$

where $c_1\in R$ is a constant as in the condition (κ) with

$$D=\{y\in Y_2\mid \|y\|_{Y_2}\leq c\}\,,\ c=c_B\,.$$

Consequently

$$\frac{[C(x_n),x_n]_{+(-)}-c_C\,\|C(x_n)\|_{+(-)}}{\|x_n\|_Y}\geq$$

$$\geq\frac{[A(x_n),x_n]_{+(-)}-c_A\,\|A(x_n)\|_{+(-)}}{\|x_n\|_Y}+$$

$$+\frac{[B(x_n),x_n]_{+(-)}-c_B\,\|B(x_n)\|_{+(-)}}{\|x_n\|_Y}\to+\infty\quad\text{as}\quad n\to+\infty,$$

and this is in contradiction with (3.10).

If $\|x_n\|_{Y_1}\leq c\ \ \forall n\geq 1$ and $\|x_n\|_{Y_2}\to+\infty$ as $n\to+\infty$ the reasoning is the same.

When $\|x_n\|_{Y_1}\to+\infty$ and $\|x_n\|_{Y_2}\to+\infty$ as $n\to+\infty$, we get the contradiction

$$+\infty>\sup_{n\geq 1}\frac{[C(x_n),x_n]_{+(-)}-c_C\,\|C(x_n)\|_{+(-)}}{\|x_n\|_Y}\geq\gamma_A(\|x_n\|_{Y_1})\frac{\|x_n\|_{Y_1}}{\|x_n\|_{Y_1}+\|x_n\|_{Y_2}}+$$

$$+\gamma_B(\|x_n\|_{Y_2})\frac{\|x_n\|_{Y_2}}{\|x_n\|_{Y_1}+\|x_n\|_{Y_2}}\geq\min\left\{\gamma_A(\|x_n\|_{Y_1}),\gamma_B(\|x_n\|_{Y_2})\right\}\to+\infty\,.$$

Proposition 10. If the multi-valued operator $A: Y \to 2^{Y^*}$ satisfies the condition (Π), then it satisfies the condition $(\kappa)_+$.

Proof. We prove this proposition arguing by contradiction. Let $D \subset Y$ be a bounded set such that for each $c > 0$ there exists $v_c \in D \setminus \{\bar{0}\}$: $[A(v_c), v_c]_+ \le -c \| v_c \|_Y \le 0$. Then due to the condition (Π)

$$\sup_{c>0} \| A(v_c) \|_+ =: d < +\infty.$$

Thus

$$-c \| v_c \|_Y \ge [A(v_c), v_c]_+ \ge - \| A(v_c) \|_+ \| v_c \|_Y \ge -d \| v_c \|_Y$$

and $(d - c) \| v_c \|_Y \ge 0$ for each $c > 0$. This is a contradiction with $v_c \ne \bar{0}$.

Proposition 11. Let the functional $\varphi : Y \to R$ be convex, lower semicontinuous on Y. Then the multi-valued map $B = \partial\varphi : Y \to C_v(Y^*)$ is λ_0-pseudomonotone on Y and it satisfies the condition (Π).

Proof. a) Property (Π). Let $k > 0$ and the bounded set $B \subset Y$ be arbitrary fixed. Then $\forall\, y \in B$ and $\forall\, d(y) \in \partial\varphi(y)$ $< d(y), y - y_0 >_Y \le k$ is fulfilled. Let $u \in Y$ be arbitrary fixed, so

$$< d(y), u >_Y = < d(y), u - y >_Y + < d(y), y >_Y \le \varphi(u) - \varphi(y) + k \le$$

$$\le \varphi(u) - \inf_{y \in B} \varphi(y) + k \equiv \text{const} < +\infty,$$

since every convex lower semicontinuous functional is lower bounded by every bounded set. Hence, thanks to the Banach-Steinhaus theorem, there exists $N = N(y_0, k, B)$ such that $\| d(y) \|_{Y^*} \le N$ for each $y \in B$;

b) λ_0-pseudomonotony on Y. Let $y_n \xrightarrow{w} y_0$ in Y, $\partial\varphi(y_n) \ni d_n \xrightarrow{w} d$ in Y^* and the inequality (2.1) true. Then, due to the monotony of $\partial\varphi$, for each $d_0 \in \partial\varphi(y_0)$ and for each $n \ge 1$

$$< d_n, y_n - y_0 >_Y = < d_n - d_0, y_n - y_0 >_Y + < d_0, y_n - y_0 >_Y \ge < d_0, y_n - y_0 >_Y .$$

Hence

$$\underline{\lim}_{n \to +\infty} < d_n, y_n - y_0 >_Y \ge \underline{\lim}_{n \to +\infty} < d_0, y_n - y_0 >_Y = 0 .$$

Because of the last inequality and of the inequality (2.1) it results in

$$\lim_{n \to +\infty} < d_n, y_n - y_0 >_Y = 0 .$$

Thus for each $w \in Y$

$$\underline{\lim}_{n \to +\infty} < d_n, y_n - w >_Y \ge \lim_{n \to +\infty} < d_n, y_n - y_0 >_Y +$$

$$+ \underline{\lim}_{n \to +\infty} < d_n, y_0 - w >_Y = < d_0, y_0 - w >_Y . \qquad (3.11)$$

From another side we have

$$< d_0, w - y_0 >_Y \le \overline{\lim_{n\to+\infty}} < d_n, w - y_n >_Y \le \varphi(w) -$$

$$- \underline{\lim_{n\to+\infty}} \varphi(y_n) \le \varphi(w) - \varphi(y_0). \qquad (3.12)$$

since every convex lower semicontinuous functional is weakly lower semicontinuous. From (3.12) it follows that $d_0 \in \partial\varphi(y_0)$. From here, due to the inequality (3.11), we obtain the inequality (2.2) as $B = \partial\varphi$ on Y.

4. EXAMPLE

Now we consider a class of w_{λ_0}-pseudomonotone maps. Let us consider the bounded domain $\Omega \subset R^n$ with rather smooth boundary $\partial\Omega$, $S = [0,T]$, $Q = \Omega \times (0;T)$, $\Gamma_T = \partial\Omega \times (0;T)$. Let as $i = 1,2$ $m_i \in R$, N_1^i (respectively N_2^i) the number of the derivatives respect to the variable x of order $\le m_i - 1$ (respectively m_i) and $\{A_\alpha^i(x,t,\eta,\xi)\}_{|\alpha|\le m_i}$ be a family of real functions defined in $Q \times R^{N_1^i} \times R^{N_2^i}$. Let

$$D^k u = \{D^\beta u, | \beta| = k\} \text{ be the differentiations by } x,$$

$$\delta_i u = \{u,\ Du,\ ...,\ D^{m_i-1}u\},$$

$$A_\alpha^i(x,t,\delta_i u, D^{m_i} v) : x,t \to A_\alpha^i(x,t,\delta_i u(x,t),\ D^{m_i} v(x,t)).$$

Moreover, let $\psi : R \to R$ be a convex, lower semicontinuous coercive real function and $\Phi : R \to C_v(R)$ be its subdifferential.

Let us assume $H = L_2(\Omega)$ and $V_i = W_0^{m_i,p_i}(\Omega)$ with $p_i \in (1,2]$ such that $V_i \subset H$ with continuous embedding, $p_i^{-1} + q_i^{-1} = 1$, $\partial\varphi(y)$ is the Gateaux subdifferential of the convex lower semicontinuous coercive functional

$$L_2(S; L_2(\Omega)) \ni y \to \varphi(y) = \int_Q \psi(y(x,t))\,dxdt$$

in the space $L_2(S; L_2(\Omega))$.

Definition of operators A_i. Let $A_\alpha^i(x,t,\eta,\xi)$, defined in $Q \times R^{N_1^i} \times R^{N_2^i}$, satisfying the conditions

$$\text{for almost each } x,t \in Q \text{ the map } \eta,\xi \to A_\alpha^i(x,t,\eta,\xi)$$

$$\text{is continuous on } R^{N_1^i} \times R^{N_2^i}; \qquad (4.1)$$

$$\text{for each } \eta,\xi \text{ the map } x,t \to A_\alpha^i(x,t,\eta,\xi) \text{ is measurable on } Q,$$

for each $u, v \in L^{p_i}(0,T;V_i) =: K_i \quad A_\alpha^i(x,t,\delta_i u, D^{m_i}u) \in L^{q_i}(Q)$. (4.2)

Then for each $u \in K_i$ the map

$$w \to a_i(u,w) = \sum_{|\alpha| \le m_i} \int_Q A_\alpha^i(x,t,\delta_i u, D^{m_i}u) D^\alpha w dx dt\,,$$

is continuous on K_i and then

there exists $A_i(u) \in K_i^*$ such that $a_i(u,w) = < A_i(u), w >$. (4.3)

Conditions on A_i. Similarly to [20, sections 2.2.5, 2.2.6, 3.2.1] we have

$$A_i(u) = A_i(u,u),\ A_i(u,v) = A_{i1}(u,v) + A_{i2}(u)\,,$$

where

$$< A_{i1}(u,v), w > = \sum_{|\alpha| = m_i} \int_Q A_\alpha^i(x,t,\delta_i u, D^{m_i}v) D^\alpha w dx dt\,,$$

$$< A_{i2}(u), w > = \sum_{|\alpha| \le m_i - 1} \int_Q A_\alpha^i(x,t,\delta_i u, D^{m_i}u) D^\alpha w dx dt\,.$$

We add the next conditions:

$$< A_{i1}(u,u), u - v > - < A_{i1}(u,v), u - v > \ge 0 \quad \forall u, v \in K_i\,, \tag{4.4}$$

if $u_j \xrightarrow{w} u$ in K_i, $u_j' \xrightarrow{w} u'$ in K_i^* and if $< A_{i1}(u_j,u_j) - A_{i1}(u_j,u), u_j - u > \to 0$,

then $A_\alpha^i(x,t,\delta u_j, D^{m_i}u_j) \xrightarrow{w} A_\alpha^i(x,t,\delta u, D^{m_i}u)$ in $L^{q_i}(Q)$, (4.5)

coercivity. (4.6)

Remark 8. Similarly to [20, theorem 2.2.8] the sufficient conditions to get (4.4), (4.5) are:

$$\sum_{|\alpha| = m_i} A_\alpha^i(x,t,\eta,\xi)\xi_\alpha \frac{1}{|\xi| + |\xi|^{p_i - 1}} \to +\infty \ \ as\ \ |\xi| \to \infty$$

for almost each $x, t \in Q$ and $|\eta|$ bounded;

$$\sum_{|\alpha| = m_i} (A_\alpha^i(x,t,\eta,\xi) - A_\alpha^i(x,t,\eta,\xi^*))(\xi_\alpha - \xi_\alpha^*) > 0 \ as\ \xi \ne \xi^*$$

for almost each $x, t \in Q$ and η.

The next condition gives the coercivity:

$$\sum_{|\alpha| = m_i} A_\alpha^i(x,t,\eta,\xi)\xi_\alpha \ge c|\xi|^{p_i} \ \text{ as rather large } |\xi|\,.$$

A sufficient condition to get (4.2) (see [20, p. 332]) is the following one:

$$|A_\alpha^i(x,t,\eta,\xi)| \le c\left[|\eta|^{p_i - 1} + |\xi|^{p_i - 1} + k(x,t)\right], \quad k \in L_{q_i}(Q)\,. \tag{4.7}$$

Arguing by analogy with the proof of [20, theorem 3.2.1] and of [20, statement 2.2.6] we get the next.

Proposition 12. ([20], p.337) Let $A_i : K_i \to K_i^*$ ($i = 1,2$) be the operator, defined in (4.3), satisfying (4.1), (4.2), (4.4), (4.5) and (4.6). Then A_i is λ-pseudomonotone on

$$W_i = \{y \in K_i | y' \in K_1^* + K_2^*\}$$

(in the classical sense) and coercive. Moreover it is bounded if (4.7) holds.

From the lemma 6 and the lemma 1 we deduce the next corollary:

Corollary 2. Let $A_i : K_i \to K_i^*$ ($i = 1,2$) be the operator, defined in (4.3), satisfying (4.1), (4.2), (4.4)–(4.7). Then $A = A_1 + A_2 : X_1 = K_1 \cap K_2 \to$ $\to K_1^* + K_2^* = X^*$ is λ_0-pseudomonotone on

$$W_1 = \{y \in X \mid y' \in X^*\}$$

and coercive.

Due to the proposition 11, to the proposition 12, to the lemma 6 and the lemma 1 it is easy to obtain the next

Corollary 3. Let $A_i : K_i \to K_i^*$ ($i = 1,2$) be the operator, defined in (4.3), satisfying (4.1), (4.2), (4.4)–(4.7); $\varphi : G = L_2(S; H) \to R$ be the functional satisfying the conditions of the proposition 11 and of the corollary 1. Then the multi-valued map

$$A = A_1 + A_2 + \partial\varphi : X \cap G \to C_v(X^* + G)$$

is $+(-)$-reflexive, λ_0-pseudomonotone on

$$W = \{y \in X \cap G \mid y' \in X^* + G\}$$

and it satisfies the condition (Π).

4.1. An application. By virtue of the corollary 3 and of the [7] (theorem 1), under the conditions of the corollary 3, the problem

$$\frac{\partial y(x,t)}{\partial t} + \sum_{|\alpha| \le m_1} (-1)^{|\alpha|} D^\alpha (A_\alpha^1(x,t,\delta_1 y, D^{m_1} y)) +$$

$$+ \sum_{|\alpha| \le m_2} (-1)^{|\alpha|} D^\alpha (A_\alpha^2(x,t,\delta_2 y, D^{m_2} y)) + \Phi(y(x,t)) \ni f(x,t) \text{ in } Q,$$

$$D^\alpha y(x,t) = 0 \quad \text{on} \quad \Gamma_T \quad \text{as} \ |\alpha| \le m_i - 1 \ \text{ and } \ i = 1,2,$$

$$y(x,0) = 0 \quad \text{in} \quad \Omega$$

has a solution in W.

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