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ON GARCH(*p,q*) CONVERGENCE

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The paper deals with symmetric GARCH(*p,q*) model. Assuming that there exists defined by this model stationary time series, we have proposed the necessary and sufficient condition for exponential mean square convergence of any stochastic recurrent procedure satisfying this model to the above stationary time series. A mathematical background of the proposal approach is based on the derived covariance method for mean square exponential stability analysis of linear stochastic difference equations, which permits one to state a mean square convergence criterion for GARCH(*p,q*) models with any integer positive p and q in the convenient for application form of an integral inequality involving the model parameters.

1. INTRODUCTION: STATIONARY GARCH(*p,q*) MODELS

Over the last decade, there has been a tendency to employ to analysis the financial time-series data model the regression equation for exogenous variables $X_t^{(s)}$, $k=1,2,...,N$ endogenous variables Y_t, and residuals U_t defined by formulae

$$Y_t = b_0 + \sum_{m=1}^{n} a_m Y_{t-m} + \sum_{m=1}^{n}\sum_{s=1}^{N} b_{ms} X_{t-m}^{(s)} + U_t,$$

$$U_t = \sum_{l=1}^{n} c_l U_{t-l} + \xi_t,\ E\{\xi_t/\Phi_{t-1}\} \equiv 0,\ E\{\xi_t^2/\Phi_{t-1}\} = \sigma_t^2, \tag{1}$$

where $\{\varepsilon_t, t\in Z\}$ is white-noise type time series (that is, i.i.d. random variables with mean zero and variance one), Φ_{t-1} is sigma-algebra of information up to time $t-1$, defined by random variables $\{\varepsilon_s,\ s\le t-1\}$, and $\{\xi_t,\ t\in Z\}$ is time series of errors (shocks) with variance, that is given as $\mathrm{GARCH}(p,q)$ process (*Generalized Auto Regressive Conditional Heteroskedasticity*), that takes the following form [4]:

$$\sigma_t^2 = \theta_0 + \sum_{k=1}^{p} \varphi_k \sigma_{t-k}^2 + \sum_{k=1}^{q} \theta_k \sigma_{t-k}^2 \varepsilon_{t-k}^2. \tag{2}$$

The above processes are defined for time moments $t\in Z$ by $q+1$ coefficients $\{\theta_0>0,\ \theta_k\ge 0,\ k=1,...,q\}$, p coefficients $\{\varphi_k \ge 0,\ k=1,2,...,p\}$, mean b_0, $nN+1$ linear regression coefficients, conditional variance σ_t^2 and distribution of

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random variable ε_0. As it has been shown by [1], under assumption $\sum_{k=1}^{p}\varphi_k + \sum_{k=1}^{q}\theta_k < 1$ there exists defined by (2) stationary time-series $\{\hat{\sigma}_t^2,\ t \in Z\}$ and expectation of deviations $u_t := \sigma_t^2 - \hat{\sigma}_t^2$ of any other satisfying (2) time series $\{\sigma_t^2,\ t \in Z\}$ converge to zero in the mean with $t \to \infty$, that is $\lim_{t\to\infty} E\,|\sigma_t^2 - \hat{\sigma}_t^2| = 0$. This paper supposes the above inequality $\sum_{k=1}^{p}\varphi_k + {}$ $+ \sum_{k=1}^{q}\theta_k < 1$ to be fulfilled. It should be mentioned that parameters of regression model (2) are mainly defined by the least square method and therefore it is preferable [5] to analyze a behavior of the second moments of iterations (2) with $t \to \infty$. We will say that the stationary GARCH model (2) is *exponential mean square stable* if the above second moments exponentially tend to zero as $t \to \infty$, that is, there exist such positive numbers M, λ that

$$\mathbf{E}\{|\sigma_t^2 - \hat{\sigma}_t^2|^2\} \le M e^{-\lambda(t-s)}\mathbf{E}\{|\sigma_s^2 - \hat{\sigma}_s^2|^2\} \qquad (3)$$

for any $t \ge s$, $s \in Z$. The problem arises: to determine a largest set of parameters of model (2), which guarantees the stability property (3). For GARCH(p,1) models this problem has been discussed in the paper [2]. Applying some of well known mathematical results for positive defined matrices, the mentioned paper derives the necessary and sufficient condition for exponential mean square stability in a form of inequality involving forth moment of ε_t and parameters $\varphi_1,\ldots,\varphi_p,\theta_1$. In spite of the convenience for application of the proposal there approach for $q=1$, that has been written as an inequality for two specially constructed determinants, it becomes very complicated for GARCH(p,q)-models with $q \ge 2$. To eliminate this lack we will apply another method, developed in paper [3] for asymptotical stability analysis of linear stochastic difference equations. It permits us to derive necessary and sufficient exponential mean square stability condition for any *p* and *q* in convenient for application form.

2. INTEGRAL CRITERIA FOR GARCH(*p,q*) EXPONENTIAL MEAN SQUARE STABILITY

It is easy to write for the deviations $u_t := \hat{\sigma}_t^2 - \sigma_t^2$ the homogeneous difference equation

$$u_t = \sum_{k=1}^{m} a_k u_{t-k} + \sum_{k=1}^{q}\theta_k u_{t-k} y_{t-k}, \qquad (4)$$

where $a_k = \varphi_k + \theta_k$, if $k \le \min\{p,q\}$, θ_k if $q < k \le p$, and φ_k if $p < k \le q$, $m = \max\{p,q\}$ and $y_t = \varepsilon_t^2 - 1$. The latter random variables $\{y_t,\ t \in Z\}$ are i.i.d.

with mean zero and variance $s^4 := E|\varepsilon_t^2 - 1|^2$ defined by distribution of ε_t. Formula (4) defines a linear difference equation with random coefficients and the problem is: to find necessary and sufficient conditions for exponential mean square decreasing of its solutions. Let sequence $\{u_t,\ t \in Z\}$ be a solution of (4). According to proposal in [3] method first of all we have to define two sequences: $\{h_t,\ t \in Z\}$, satisfying for $t > 0$ homogeneous difference equation $h_t = a_1 h_{t-1} + a_2 h_{t-2} + ... + a_m h_{t-m}, (h)$ under conditions $h_0 = 1,\ h_t = 0$ for $t \le -1$, and $\{\tilde{x}_t, t > 0\}$ satisfying the same homogeneous difference equation $\tilde{x}_t = a_1 \tilde{x}_{t-1} + a_2 \tilde{x}_{t-2} + ... + a_m \tilde{x}_{t-m}$, but for $t \le 0$ is the same as u_t, that is, $\tilde{x}_t = u_t, t \le 0$. Now we should rewrite equation (4) in a following form $u_t = g_t + \sum_{j=1}^{q} \sum_{k=1}^{t} h_{t-k-j} \theta_j y_k u_k$, where $g_t = \tilde{x}_t + \sum_{j=1}^{q} \sum_{s=1}^{j} h_{t-s} \theta_j y_{s-j} u_{s-j}$ is Φ_0-adopted random sequence for any $t \ge 0$. Squaring the both parts of the above equity and taking a conditional expectation under condition Φ_0 we can reach for conditional second moment $m_t := E\{|u_t|^2 / \Phi_0\}$ an equation $m_t = g_t^2 + s^4 \sum_{k=1}^{t} b_{t-k}^2 m_k$, where $b_t = \sum_{j=1}^{q} h_{t-j} \theta_j$. Because g_t^2 and b_t^2 are exponentially decreasing to zero nonnegative sequences, any satisfying (4) positive sequence $\{m_t,\ t \ge 0\}$ may be majorized by sequence $\{c^t,\ t \ge 0\}$ for sufficiently large c. Therefore to analyze an asymptotic of this sequence we may apply discrete Laplace transformation multiplying the both parts of equation for m^t by z^t with some constant $z \in (0, c^{-1})$ and summarizing by t from 0 to ∞. This approach permits to find function $M(z) := \sum_{t=0}^{\infty} z^t m_t$ in a form of fraction $M(z) = G(z)/(1 - s^4 B(z))$, where $G(z) := \sum_{t=0}^{\infty} z^t g_t$, $B(z) := \sum_{t=0}^{\infty} z^t b_t^2$. It is obviously that m_t exponentially decreases with $t \to \infty$ if and only if the series $\sum_{t=0}^{\infty} m_t$ converges. Therefore one can make sure of equivalence the latter assertion to inequality $E\{|\varepsilon_t|^2 - 1|^2\} < (B(1))^{-1}$ involving fourth moment of white noise and parameters of GARCH(p,q). Let $B_1(z)$ be a discrete Laplace transformation of sequence $\{b_t\}$, that is, $B_1(z) := \sum_{t=0}^{\infty} b_t z^t$. Applying the well known Cauchy theorem one can find

$$B(1) = \frac{1}{2\pi i} \int_{|z|=1} \frac{\sum_{j=k}^{q} \sum_{k=1}^{q} \theta_j \theta_k z^{j-k-1}}{\left(1 - \sum_{k=1}^{m} a_k z^k\right)\left(1 - \sum_{k=1}^{m} a_k z^{-k}\right)} dz .$$

The function $B_1(z)$ is a Z-transformation of series $b_t = \sum_{j=1}^{q} h_{t-j}\theta_j$. Therefore applying Z-transformation one can find expression $B(1)$ in an integral form

$$S_4^{-1} = \frac{1}{2\pi i} \int_{|z|=1} \frac{\left(\sum_{j=k}^{q} \sum_{k=1}^{q} \theta_j \theta_k z^{j-k} \right) z^{m-1}}{\left(1 - \sum_{k=1}^{m} a_k z^k \right)\left(1 - \sum_{k=1}^{m} a_k z^{m-k} \right)} dz , \qquad (5)$$

where $m = \max\{p,q\}$ and a_k defined above in formula (a_k). Therefore the necessary and sufficient condition for stationary GARCH (p,q) mean square stability has a form an inequality $\mathrm{E}\varepsilon_t^4 < 1 + S_4$. The integral in (5) can be calculated applying residual theory. For example necessary and sufficient exponential mean square stability condition of GARCH $(2,2)$ has following complete form:

$$\mathrm{E}\varepsilon_t^4 < 1 + \frac{[(1-\varphi_2-\theta_2)^2 - (\varphi_1+\theta_1)^2](1+\varphi_2+\theta_2)}{(\theta_1^2+\theta_2^2)(1-\varphi_2-\theta_2) + 4(\varphi_1+\theta_1)\theta_1\theta_2}.$$

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