UDK 517.9

# TO THE QUESTION OF MIXED TYPE SYSTEM SIMULATION IN THE TASKS OF ANALYSIS AND CONTROL

**G. IOVANE, V.M. MIZERNYY**

The paper presents the research of the mathematical models of mixed systems and considers the principal tasks of analysis, controlling and evaluation of objects' states parameters, described by nonlinear integral and differential equations with partial derivatives.

## INTRODUCTION

In the context of making tasks on simulation and controlling for complex systems, which arise in different branches of physics, chemistry, economics and so on, different approaches were developed based upon the conception of mixed systems [1–4]. The number of control processes in mathematical models, which contain systems of equations of different types, as for instance differential and integral equations, contain blocks with distributed and concentrated parameters, multivariable and discrete-continued systems [5–6], etc. The progress of research of mathematical modelling for objects with distributed parameters is due to link to the great development of nonlinear analysis method, which is applicable in different spheres of mathematics [7–9]. Thus, it is quite natural to reduce the study of these models to nonlinear operator, differential-operator equations, variable inequalities and systems, which contain the above objects. With this approach, the results for specific objects will be the consequence of operator methods.

## TASK SETTING

For the description of some nonstationary processes, which take place in the dimensional sphere $\Omega \subset R^N$ during the time $S$, we operate with functions of time and coordinates, that is, with the function $z$, which brings the actual number of vector $z(t,\omega)$ to conformity with each pair $(t,\omega) \subset S \times \Omega$. The variables $t$ and $\omega$ are independent.

Another convenient approach to the mathematical description of nonstationary processes allows to work with functions, which bring the coordinate function $z(t,\cdot)$ determined on $S$ to conformity with each moment of time $t$, with the determination in some space $Z$, that is $z \in (S \to Z)$.

Let us consider some nonstationary tasks, whose description is made with the help of nonlinear functional equations system.

**Task 1.** Let $\Omega$ be the restricted sphere of $R^N$ with regular bound $\partial\Omega$ and the $S$ time interval $S=[0,T]$, $T>0$; we consider

$$x(t,\omega)+\int_{\Omega}K(\omega,w)\,h(w,(z(t,w),x(t,w)))dw=g(t,w)\,, \tag{1}$$

$$\frac{\partial z}{\partial t}(t,\omega)-\sum_{i,j=1}^{N}\frac{\partial}{\partial\omega_i}\left(a_{ij}\,\frac{\partial z}{\partial\omega_j}(t,\omega)\right)+Q(\omega,x(t,\omega),z(t,\omega))=f(t,\omega) \tag{2}$$

$$\forall(t,\omega)\in S\times\Omega,\quad z(0,\omega)=\gamma(\omega),\quad z(t,x)\Big|_{\Sigma}=0\quad\forall t\in S, \tag{3}$$

where $\Sigma=\partial\Omega\times S$, the factors $a_{ij}$ are constant values.

Through $z(t,\cdot),\ t\in S$ we label a function, designated on $S\times\overline{\Omega}$, which has the fixed variable $t$.

The classical solution for the system (1)–(3) is function of $x(t,\omega),\ z(t,\omega)$, designated on $S\times\overline{\Omega}$; and the function $z(t,\omega)$ has to be continuously differentiable with respect to $t$ and double differentiable with respect to $\omega$, and should fulfil the conditions (3). The function $z(t,\omega)$ is differentiable with respect to $\omega$.

Typically, proving the theorems of existence for classical solution of tasks (2)–(3) requires the application of complicated mathematic technique. That is why the proceeding from the classical task (1)–(3) to the corresponding task in functional and analytical setting is quite logical. Consequently, let us introduce some functional spaces with $(S\to Z)$ [5] and the following symbols:

$$y(t)=x(t,\cdot);\quad \psi(t)=z(t,\cdot)\,,$$

$$By(t)=(Dy)(t,\cdot);\ (Dy)(t,\omega)=\int_{\Omega}K(\omega,w)\,x(t,w)dw\,,$$

$$F\big(\psi(t),y(t)\big)=h\big(\cdot,z(t,\cdot),x(t,\cdot)\big);\ \ b(t)=g(t)\,,$$

$$G(y(t),\psi(t))=Q(\cdot,x(t,\cdot),z(t,\cdot))\,,$$

$$L\psi(t)=Ez(t,\cdot);\ \ (Ez)(t,\omega)=-\sum_{i,j=1}^{N}\frac{\partial}{\partial\omega_i}\left(a_{ij}\,\frac{\partial z}{\partial\omega_j}(t,\omega)\right), \tag{4}$$

$$\psi'(t)=\frac{\partial z}{\partial t}(t,\cdot);\quad \psi(0)=\gamma(\cdot);\quad \varphi(t)=f(t,\cdot)\,.$$

Considering the above relations, the system (1)–(3) with the initial and boundary conditions can be represented in the terms of operator equations:

$$y(t)+FB(\psi(t),y(t))=b(t)\,, \tag{5}$$

$$\psi'(t)+L\psi(t)+G(y(t),\psi(t))=\varphi(t)\,, \tag{6}$$

$$\psi(0)=\gamma(\cdot)\,. \tag{7}$$

Let us assume, that the abstract functions belong to classes

$$y(t) \in (S \to L_p(\Omega)),$$

$$\psi(t) \in (S \to W_p^m(\Omega)); \ \psi(t) \in \left( S \to \left[ W_p^m(\Omega) \right]^* \right),$$

and the operators, which belong to (5)–(6), act according to the rules:

$$F : W_p^m(\Omega) \times L_p(\Omega) \to L_q(\Omega),$$

$$B : L_p(\Omega) \to L_q(\Omega),$$

$$L : W_p^m(\Omega) \to \left[ W_p^m(\Omega) \right],$$

$$G : L_p(\Omega) \times W_p^m(\Omega) \to \left[ W_p^m(\Omega) \right],$$

where $G$ and $F$ are nonlinear reflections, $L$, $B$ are linear, $L_p(\Omega)$ is the space of p-integrable functions, $W_p^m(\Omega)$ is Sobolev space [5], $\left[ W_p^m(\Omega) \right]^*$ is the integrated space.

We note that the operators $F$, $B$, $L$, $G$ do not depend explicitly on variable $t$.

**Task 2.** Instead of the system (1)–(3), we consider the following system:

$$x(t,\omega) + \int_{\Omega} K(\omega, w)\, h(w, (z(t,w), x(t,w)))dw = g(t,w), \tag{8}$$

$$\frac{\partial z}{\partial t}(t,\omega) - \sum_{i,j=1}^{N} \frac{\partial}{\partial \omega_i}\left( a_{ij} \frac{\partial z}{\partial \omega_j}(t,\omega) \right) + Q(\omega, x(t,\omega), z(t,\omega)) = f(t,\omega) \tag{9}$$

$$\forall\, (t,\omega) \in S \times \Omega; \ z(0,\omega) = \gamma(\omega); \ z(t,x)\big|_{\Sigma} = 0 \quad \forall\, t \in S, \tag{10}$$

where the factors $a_{ij}$ are constant values.

We introduce the following functions and expressions:

$$y(t) = x(t,\cdot); \ \psi(t) = z(t,\cdot); \ \psi'(t) = \frac{\partial z}{\partial t}(t,\cdot),$$

$$B(t)\, y(t) = D(t,\cdot) Y(t,\cdot); \quad D(t,\omega) Y(t,\omega) = \int_{\Omega} K(\omega, \cdot, w)\, x(t,w) dw,$$

$$F(t)(\psi(t), y(t)) = h(t, \cdot, z(t,\cdot), x(t,\cdot)); \ b(t) = g(t,\cdot),$$

$$G(t)(y(t), \psi(t)) = Q(t, \cdot, x(t,\cdot), z(t,\cdot)),$$

$$L\psi(t) = (Ez)(t,\cdot), (Ez)(t,\omega) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial \omega_i}\left( a_{ij} \frac{\partial z}{\partial \omega_j}(t,\omega) \right),$$

$$\psi(0) = \gamma(\cdot); \quad \varphi(t) = f(t,\cdot)$$

and from (8)–(10) we proceed to the system of operator equations:

$$y(t) + B(t)F(t)(\psi(t), y(t)) = b(t)\,, \qquad (11)$$

$$\psi'(t) + L\psi(t) + G(t)(y(t), \psi(t)) = \varphi(t)\,, \qquad (12)$$

$$\psi(0) = \gamma\,, \qquad (13)$$

where the operators, which act on abstract functions, depend on variable $t$ explicitly. In detail, $B(t)$, $F(t)$, $G(t)$ act in the spaces, determined by the task (5)–(7).

In the given example $Z' \in (S \to Z^*)$ shall be understood as follows: the derivative from $Z$ with the respect to $t$ in the sense of space devision $D^*(S, Z^*)$ can be represented with the help of the function $(S \to Z^*)$ [5].

Task (1)–(3) or (8)–(10) with the initial and boundary conditions and the corresponding tasks (5)–(7) and (11)–(13) are equivalent in some sense.

This can be proved with the application of such lemma:

**Lemma 1** [5]. The following relation

$$\psi(t) = z(t, \cdot) \quad \forall t \in S \qquad (14)$$

determines the mutual correspondence in a unique fashion $Z \to \psi$ between the function $z \in C(S \times \overline{\Omega})$ and the function $\psi \in C(S; C(\overline{\Omega}))$. The function $z \in C(S \times \overline{\Omega})$ has partial derivative $\frac{\partial z}{\partial t} \in C(S \times \overline{\Omega})$ and $\frac{\partial z}{\partial t}(t, \cdot) = \psi'(t)$ for each a $t \in S$.

Let us consider the conditions for a possible realization of the models of mixed objects, which are described by system of nonlinear integer equations and evolution equations with integro-differential operators.

Let $\Omega$ be a restricted sphere in Euclidean space $R^N$ and $X$, $Z$ be Banach spaces of functions on $\Omega$ with the norm $\|\cdot\|_X$, $\|\cdot\|_Z$ and $S = [0, T]$, $T > 0$.

Let us consider the equation system

$$x(t, \omega) + \int_\Omega K(\omega, w)\, h(t, w, z(t, w), x(t, w))dw = g(t, \omega)\,, \qquad (15)$$

$$\frac{\partial z(t,\omega)}{\partial t} - \int_\Omega L(\omega, w) Q\left( t, \omega, x(t, w), z(t, w), \frac{\partial z}{\partial w_1}, \ldots, \frac{\partial z}{\partial w_N}, \frac{\partial^2 z}{\partial w_1^2}, \ldots, \frac{\partial^2 z}{\partial w_N^2} \right) dw =$$

$$= p(t, \omega) \qquad (16)$$

$$\forall (t, \omega) \in S \times \Omega,\ \ z(0, \cdot) = \mathrm{Z}_0 \in Z,\ \ z(t, \omega)\big|_\Sigma = 0,\ \ \Sigma = (0, \mathrm{T}) \times \partial\Omega\,, \qquad (17)$$

which we will write [5] in the kind of operator

$$y(t) + BF(t)(\psi(t), y(t)) = \varphi(t)\,, \qquad (18)$$

$$\psi'(t) + G(t)(y(t), \psi(t)) = f(t) \qquad \forall t \in \mathrm{S} \qquad (19)$$

with the initial condition $\psi(0) = \gamma$.

We stress that in the first equation the variable *t* acts as a parameter.

Here $F=\{F(t)\}$, $G=\{G(t)\}$, $t\in S$ are in the families of nonlinear operators, acting in spaces $F(t): Z\times X\to X^*$, $G(t): X\times Z\to Z$ $\forall t\in \mathrm{S}$, and the operator $B: X^*\to X$ is a linear one. The functions $t\to y(t)$ and $t\to\psi(t)$ are determined for $t\in S$ and belong to spaces $C(S;X)$ and $C(S;Z)$ accordingly.

Let us assume that the family of operators $F=\{F(t)\}$, $G=\{G(t)\}$ satisfies the following conditions:

a) for each a $y\in X$ and $\psi\in C(S;Z)$, the function $S\ni t\mapsto F(t)(\psi(t),y)\in$ $\in X^*$ is of class $C(S;X^*)$;

b) for each a $z\in Z$ and $h\in C(S;X)$, the function $S\ni t\mapsto G(t)(h(t),z)\in$ $\in Z$ is of class $C(S;Z)$;

c) the operators $F(t)(\psi,\cdot)\in(X\to X^*)$ and $G(t)(y,\cdot)\in(Z\to Z)$ are equally continuous, that is, there are the constants $r_1$ and $r_2$, which are independent from $t$, and follow the condition

$$\left\|F(t)(\psi,y_1)-F(t)(\psi,y_2)\right\|_{X^*}\le r_1\left\|y_1-y_2\right\|_{\mathrm{X}}\quad \forall\psi\in \mathrm{Z},\ \forall y_1,y_2\in X$$

and

$$\left\|G(t)(y,\psi_1)-G(t)(y,\psi_2)\right\|_Z\le r_2\left\|\psi_1-\psi_2\right\|_Z\quad \forall y\in X,\ \forall\psi_1,\psi_2\in Z.$$

Let us assume that $G(y,t)=\{G(t)(y(t),\psi(t))\}$.

**Lemma 2**. Let the family of operators $G=\{G(t)\}$ satisfy the conditions b), c) $\forall y\in C(S;X)$ and $\psi\in C(S;Z)$.

Then $G(y,\psi)\in C(S;Z)$.

**Proof.** Let $\{t_n\}\subset S$ be any sequence, with $t_n\to t_0$ when $n\to\infty$. By virtue of execution of condition c)

$$\left\|G(t_n)(y(t_n),\psi(t_n))-G(t_0)(y(t_0),\psi(t_0))-\right.$$
$$\left.-G(t_n)(y(t_n),\psi(t_0))+G(t_n)(y(t_n),\psi(t_0))\right\|_Z\le$$
$$\le r_2\left\|\psi(t_n)-\psi(t_0)\right\|_Z+\left\|G(t_n)(y(t_n),\psi(t_0))-G(t_0)(y(t_0),\psi(t_0))\right\|_Z.$$

When $t_n\to t_0$ the augend in the right summand tends to zero due to the continuous function $\psi$, and the addend tend to zero according to the condition b).

This proves for lemma.

The analogical lemma can be formed as for the family of operators $F=\{F(t)\}$.

**Lemma 3** [5]. The norm

$$\left\|\psi\right\|_{(C,k)}=\sup_{t\in S}\left\{e^{-kt}\left\|\psi(t)\right\|_Z\right\},\quad k\ge 0, \tag{20}$$

is equivalent to the norm

$$\|\psi\|_{C(S;Z)} = \sup_{t\in S} \|\psi(t)\|_Z .$$

**Theorem 1.** Let the conditions a)÷c), related to the family of the operators $F$, $G$, come true and the operator $B$ be linear and continuous, whose norm satisfies the inequality

$$\|B\| < \frac{1}{r_1}, \text{ where } r_1 = \text{const} \text{ is constant.}$$

Then the task

$$y(t) + BF(t)(\psi(t), y(t)) = \varphi(t),$$

$$\psi'(t) + G(t)(y(t), \psi(t)) = f(t)$$

with the initial condition $\psi(0) = \gamma$ has solution for any $\varphi \in C(S;Z)$, $f \in (S;Z)$ and $\gamma \in Z$.

**Proof.** We obtain the previous result by using the principle of fixed point. Integrating the second system equation on the interval $[0;t]$, we obtain

$$\psi(t) = \gamma - \int_0^t [G(s)(y(s), \psi(s)) - f(s)] ds . \quad (21)$$

The integral here is considered in the sense of Bohner. We designate

$$(U\psi)(t) = \gamma - \int_0^t [G(s)((s), \psi(s)) - f(s)] ds , \quad (22)$$

$$U_0(t)y(t) = \varphi(t) - BF(t)(\psi(t), y(t)) . \quad (23)$$

By virtue of lemma 2 and differentiality of an indeterminate integral of Bohner [5], the operator $U$ acts from $C(S;X)\times C(S;Z)$ into $C^1(S;Z)$.

We show that with each $y \in C(S;X)$ the reflection of $U$ with some $k \geq 0$ is compressed into $(C;k)$ — to the norm of the space $C(S;Z)$.

According to c), from (22) it follows that for any $\psi_1, \psi_2 \in C(S;Z)$

$$\|(U\psi_1)(t) - (U\psi_2)(t)\|_Z \leq \int_0^t \|G(s)(y(s), \psi_1(s)) -$$

$$- G(s)(y(s), \psi_2(s))\|_Z \cdot e^{-ks} e^{ks} ds \leq r_2 \int_0^t \|\psi_1 - \psi_2\|_Z e^{-ks} e^{ks} ds \leq$$

$$\leq r_2 \|\psi_1 - \psi_2\|_{(C,k)} \left( \frac{e^{kt} - 1}{k} \right).$$

Then

$$\|(U\psi_1)(t) - (U\psi_2)(t)\|_Z e^{-kt} \leq \frac{r_2}{k}(1 - e^{-kt}) \|\psi_1 - \psi_2\|_{(C,k)} \leq$$

$$\leq \frac{r_2}{k}(1 - e^{-kt}) \|\psi_1 - \psi_2\|_{(C,k)} .$$

Taking in the left part the upper bound on the $t \in S$, we obtain

$$\|U\psi_1 - U\psi_2\|_Z \le \frac{r_2}{k}(1-e^{-kt})\|\psi_1 - \psi_2\|_{(C,k)}.$$

If we choose $k \ge r_2$, the reflection of $U$ will be compressed. So for each $y \in C(S;X)$ there is the element $\psi_0 \in C(S;Z)$, which is fixed point of reflection $U$, that is $\psi_0 = U\psi_0$.

By the virtue of (22) we have

$$\psi_0(t) = \gamma - \int_0^t [G(s)(y(s),\psi(s)) - f(s)]ds \quad \forall t \in S. \tag{24}$$

Considering that the right part of this expression has the continuous derivative on *t*, then $\psi_0 \in C^1(S;Z)$ and

$$\psi_0(t) + G(t)(y(t),\psi_0(t)) = f(t) \quad \forall t \in S,$$

and $\psi_0(0) = \gamma(\omega)$.

Then we consider the operator $U_0$ with the fixed $\psi$ . According to c) from (23) it follows that for $\forall y_1, y_2 \in C(S;X)$

$$\|U_0(t)y_1(t) - U_0(t)y_2(t)\|_X = \|BF(t)(\psi(t),y_1(t)) - BF(t)(\psi(t),y_2(t))\|_X =$$

$$= \|B[F(t)(\psi(t),y_1(t)) - F(t)(\psi(t),y_2(t))]\|_X \le \|B\| \cdot \|F(t)(\psi(t),y_1(t)) -$$

$$- F(t)(\psi(t),y_2(t))\|_{X^*} \le \|B\| \cdot r_1 \cdot \|y_1(t) - y_2(t)\|_X.$$

If

$$\|B\| \cdot r_1 < 1, \tag{25}$$

then the operator $U_0$ will be compressed in space $C(S;X)$ with each $\psi \in C(S;Z)$.

In the system of equations,

$$y(t) = \varphi(t) - BF(t)(\psi(t), y(t)), \tag{26}$$

$$\psi(t) = \gamma - \int_o^t [G(s)(y(s),\psi(s)) - f(s)]ds, \tag{27}$$

which is received from task (18)–(19) with the initial condition $\psi(0) = \gamma$, assuming that $y = y_1$, $\psi = \psi_1$, where $(y_1;\psi_1)$ — is any pair from $C(S;X) \times C(S;Z)$, we have

$$y_2(t) = \varphi(t) - BF(t)(\psi_1(t), y_1(t)).$$

Then we substitute the pair $(y_2;\psi_2)$ into the equation (13). Consequently, we get

$$\psi_2(t) = \gamma - \int_0^t [G(s)(y_2(s),\psi_1(s)) - f(s)]ds.$$

We substitute the pair $(y_2;\psi_2)$ into (26). Having determined $y_3$, we substitute the pair $(y_3;\psi_2)$ into (27). Then we determine $\psi_3$. By repeating the previous procedure, we get an iteration process

$$y_{n+1}(t) = \varphi(t) - BF(t)(\psi_n(t), y_n(t)) \tag{28}$$

and

$$\psi_{n+1}(t) = \gamma - \int_0^t \left[G(s)(y_n(s), \psi_n(s)) - f(s)\right] ds \,. \tag{29}$$

We prove that the succession $\{y_n\}, \{\psi_n\}$ converge to the fixed point $(y_0;\psi_0)$, which is the solution of the system (26)–(27) and, as a consequence, of the system (18)–(19).

With any $n$ we have

$$\|y_{n+1}(t) - y_n(t)\|_X = \|U_0(t)y_n(t) - U_0(t)y_{n-1}(t)\|_X =$$

$$= \|BF(t)(\psi_n(t), y_n(t) - BF(t)(\psi_{n-1}(t), y_{n-1}(t)))\|_X =$$

$$= \|B\left[F(t)(\psi_n(t), y_n(t)) - F(t)(\psi_{n-1}(t), y_{n-1}(t))\right]\|_X \le$$

$$\le \|B\| \cdot \|F(t)(\psi_n(t), y_n(t)) - F(t)(\psi_{n-1}(t), y_{n-1}(t))\|_{X^*} \le$$

$$\le \|B\| r_1 \|y_n(t) - y_{n-1}(t)\|_X = \alpha \|y_n(t) - y_{n-1}(t)\|_X \,,$$

where $\alpha = \|B\| r_1 < 1$.

Generally, the following chain of inequalities holds:

$$\|y_{n+1}(t) - y_n(t)\| \le \alpha \|y_n(t) - y_{n-1}(t)\|_X \le \ldots \le \alpha^{n-1} \|y_2(t) - y_1(t)\|_X \,.$$

We show that the sequence $\{y_n\}$ is fundamental for $C(S;X)$. Using the inequality of triangle and previous inequalities with $m>n$, we have

$$\|y_m - y_n\|_{C(S;X)} = \|y_m - y_{m-1} + y_{m-1} - y_{m-2} + y_{m-2} - \ldots + y_{n+1} - y_n\|_{C(S;X)} \le$$

$$\le \|y_m - y_{m-1}\|_{C(S;X)} + \|y_{m-1} - y_{m-2}\|_{C(S;X)} + \ldots + \|y_{n+1} - y_n\|_{C(S;X)} \le$$

$$\le \alpha^{m-2} \|y_2 - y_1\|_{C(S;X)} + \alpha^{m-3} \|y_2 - y_1\|_{C(S;X)} + \ldots + \alpha^{n-1} \|y_2 - y_1\|_{C(S;X)} =$$

$$= (\alpha^{m-2} + \alpha^{m-3} + \ldots + \alpha^{n-1}) \|y_2 - y_1\|_{C(S;X)} .$$

We designate the expression between brackets through

$$\Sigma_1 = \alpha^{n-1} + \alpha^n + \ldots + \alpha^{m-3} + \alpha^{m-2} .$$

Analogically through

$$\Sigma_2 = \alpha^{n-1} + \alpha^n + \ldots + \alpha^{m-3} + \alpha^{m-2} + \alpha^m + \ldots .$$

we designate the infinite sum.

Then

$$\Sigma_1 < \Sigma_2 = \Sigma_1 + \alpha^{m-1} + \alpha^m + \ldots,$$

where $\Sigma_2$ is the sum of members of infinite decreasing geometric progression, whose first member is $\alpha^{n-1}$ with the denomination $\alpha < 1$.

As well known,

$$\Sigma_2 = \frac{\alpha^{n-1}}{1-\alpha},$$

then

$$\|y_m - y_n\|_{C(S;X)} \le \Sigma_1 \|y_2 - y_1\|_{C(S;X)} \le \Sigma_2 \|y_2 - y_1\|_{C(S;X)} \le$$

$$\le \frac{\alpha^{n-1}}{1-\alpha}\|y_2 - y_1\|_{C(S;X)}.$$

Considering that with $n \to \infty$ the value in the right part of inequality tends to zero with any $m > n$, then $\|y_m - y_n\|_{C(S;X)} \to 0$ thus, the consequence $\{y_n\}$ is fundamental. In Banach space $C(S;X)$ the fundamental consequence $\{y_n\}$ has the limit $y_0$. In analogical way, we obtain the sequence $\{\psi_n\}$, fundamental in $C(S;X)$ with the limit $\psi_0$.

Proceeding to the limit $n \to \infty$ in the equations (28)–(29) and considering the condition a) (it needs in the condition of the given theorem), and using the fact that the operator $B$ in the first equation and integral operator in the second equation are linear and continuous, we obtain:

$$y_0 = \varphi(t) - BF(t)(\psi_0(t), y_0(t)),$$

$$\psi_0 = \gamma - \int_0^t [G(s)(y_0(s), \psi_0(s)) - f(s)]ds.$$

Thus, the theorem is proved.

Let us consider the system of integro-differential equations of the following type

$$x(t,\omega) + \int_\Omega K(t,\omega,w)\, h(t,w,z(t,w),x(t,w))dw = g(t,\omega), \tag{30}$$

$$\frac{\partial z}{\partial t}(t,\omega) - \int_\Omega L(\omega,w) Q\begin{pmatrix} t, w, x(t,w), z(t,w), \dfrac{\partial z}{\partial w_1}, \ldots \\ \ldots, \dfrac{\partial z}{\partial w_N}, \dfrac{\partial^2 z}{\partial w_1^2}, \ldots \dfrac{\partial^2 z}{\partial w_N^2} \end{pmatrix} dw = p(t,\omega) \tag{31}$$

$$\forall (t,\omega) \in S \times \Omega, \quad z(0,\cdot) = \gamma \in Z, \quad z(t,\omega)\big|_\Sigma = 0, \tag{32}$$

or in the operator form

$$y(t) + B(t)F(t)(\psi(t), y(t)) = \varphi(t), \tag{33}$$

$$\psi'(t) + G(t)(y(t), \psi(t)) = f(t) \quad \forall t \in S \tag{34}$$

with the initial condition $\psi(0)=\gamma$ .

We stress that in the first equation the variable $t$ acts as a parameter.

The family of nonlinear operators

$$F=\{F(t)\},\ \ G=\{G(t)\},\ \ t\in S$$

are the same as in (4)–(5).

$$B(t): X^* \to X \text{ explicitly depend on } t\in S .$$

Let $F$ and $G$ satisfy the conditions a)÷c), and the family of operators $B(t)$ is such that $\|B(t)\|<\frac{1}{r_1}$ $\forall t\in S$ , where the constant $r_1$ comes from condition c).

For the system (33)–(34) with the initial condition $\psi(0)=\gamma$ such statement comes true.

**Theorem 2.** Let us assume that the conditions a)÷c) come true as for the family of operators $F(t)$, $G(t)$ and the norm of the family of linear operators $B(t)$ satisfy the inequality $\|B(t)\|<\frac{1}{r_1}$ $\forall t\in S$ , where $r_1=\text{const}$ from the condition c). Then the differential-operator system

$$y(t)+B(t)F(t)(\psi(t),y(t))=\varphi(t)\,,$$

$$\psi'(t)+G(t)(y(t),\psi(t))=f(t)$$

with initial condition $\psi(0)=\gamma$ has a solution in $C(S;X)\times C(S;Z)$ for any $\varphi\in C(S;X),\ \ f\in C(S;Z)$ and $\gamma\in Z$ .

The proof of theorem 2 is obtained by the analogy with theorem 1.

Let us consider the conditions of possible realization of mixed objects models, which contains evolution equations of the second order.

In the restricted domain $\Omega\subset R^n$ with the bound $\partial\Omega$ on the interval $S=(0,T)$ , the system of equations is searched

$$x(t,\omega)+\int_\Omega K(\omega)\,h\big(t,w,z(t,w),x(t,w)\big)dw=g(t,\omega)\,, \tag{35}$$

$$\frac{\partial^2 z(t,\omega)}{\partial t^2}-\int_\Omega L(\omega,w)Q\left(t,\omega,x(t,w),z(t,w),\frac{\partial z}{\partial w_1},\ldots\right.$$
$$\left.\ldots,\frac{\partial z}{\partial w_N},\frac{\partial^2 z}{\partial w_1^2},\ldots,\frac{\partial^2 z}{\partial w_N^2}\right)dw=p(t,\omega), \tag{36}$$

where $(t,\omega)\in S\times\Omega$ , with initial and boundary conditions

$$z(0,\omega)=\gamma_0(\omega),\ \ \frac{\partial z}{\partial t}(0,\omega)=\gamma_1\omega, \tag{37}$$

$$z(t,\omega)\big|_\Sigma=0,\ \ \Sigma=S\times\partial\Omega\,. \tag{38}$$

In the first equation the variable $t$ acts as a parameter. Through $X$, $Z$ we designate the true functional spaces and consider the families $F = \{F(t),\ t \in S\}$ $G = \{G(t),\ t \in S\}$ of nonlinear operators, which act in spaces $F(t): Z \times X \to X^*$, $G(t): X \times Z \to Z$. As in the previous cases we rewrite the task (35)–(38) in operator form

$$y(t) + B(t)F(t)\big(\psi(t),\ y(t)\big) = \varphi(t)\,, \tag{39}$$

$$\psi''(t) + G(t)\big(y(t),\ \psi(t)\big) = f(t) \quad \forall t \in S \tag{40}$$

with the initial conditions

$$\psi(0) = \gamma, \quad \psi'(0) = \gamma_1, \tag{41}$$

where $B(t): X^* \to X$ is the family of linear reflections.

Task (39)–(40) with initial conditions (41) can be brought to the task, which was considered in the theorem 2, by introducing designation $\psi'(t) = q(t)$. Here instead of the families of the operators $F(t)$, $G(t)$, $B(t)$, which act as $F(t): X \to X^*$, $B(t): X^* \to X$, $G(t): Z \to Z$, we consider the following operators $G: (S \to Z) \to (S \to Z)$ з $\big(L_p(S;Z) \to L_q(S;Z)\big)$, $p > q$, $1/p + 1/q = 1$.

Thus, we get more common task setting, considering that each family $\{G(t)\}$ of operators with $(Z \to Z)$ can be given the correspondent one trajectory operator $G \in \big((S \to Z) \to (S \to Z)\big)$ according to rule $(Gz)(t) = G(t)\,z(t) \quad \forall t \in S$.

Not each operator $G: (S \to Z) \to (S \to Z)$ are interesting for our purpose. An example is given by the operators, which contain the so-called Volterra's operators a $G \in \big((S \to Z) \to (S \to Z)\big)$ (which play an important role in different practical appendixes). They are characterized by the fact that the value $(Gz)(t)$ can depend upon the values of function $z$ in the interval $[0;t]$, that is on "previous history".

Considering the above question, let us give more modified task setting. Let $Z$ be a reflective Banach space, continually and snugly put into the Hilbert space $H$ and the operator of Volterra $G$ works as

$$G: L_p(S;Z) \to L_q(S;Z^*),\ p > 1,\ 1/p + 1/q = 1,\ \ f \in L_q(S;Z^*)\,.$$

Then the following task setting will be right: $\psi' + G\psi = f$, $\psi(0) = \gamma \in H$.

Indeed, considering that $Z \subset H \subset Z^*$ (see [5], paragraph 6, ch. 1), from $\psi \in L_p(S;Z)$ it follows that $\psi \in D^*(S;Z^*)$. That is why the equation $\psi' + G\psi = f$ can be understood as the equation in $D^*(S;Z^*)$. If $\psi \in L_p(S;Z)$ satisfies the given equation, then $\psi' \in L_q(S;z^*)$ (see [5], theorem 1.17, ch. 4) $\psi \in C(S;H)$, that is the initial condition $\psi(0) = \gamma \in H$ makes sense. The results of theorems 1, 2 can be generalized for the specific case, when instead of operators from family $\{G(t)\},\ t \in S$ there are the operators of Volterra's type. The meaning of this operator can be specified more wider than in the previous considerations.

**Denotation 1** [5]. Let $Z_1, Z_2$ be linear spaces and $S = [0;T]$, $T > 0$. The reflection $G \in (D(G) \to (S \to Z_2))$, $D(G) \subset (S \to Z_1)$ is called operator of Volterrif from the equation $\psi(s) = \varphi(s)$ almost for all $s \in [0;t]$, $t \in S$, it follows that $(G\psi)(s) = (G\varphi)(s)$.

Firstly, we consider operators of Volterra, which reflect the space $C(S;Z)$ into themselves. For this case in denotation 1 it needs to put $Z_1 = Z_2 = Z$ and $D(G) = C(S;Z)$, and the expression "almost for all" change for "for all".

In this case the condition of Lipshitz for the operator of Volterra $G$ looks like:

$$\forall\, \psi_1, \psi_2 \in C(S;Z)$$

$$\|G\psi_1 - G\psi_2\|_{C(S;Z)} \le r_2 \|\psi_1 - \psi_2\|_{C(S;Z)}, \quad r_2 = \text{const}\,. \tag{42}$$

As for the system of operator equations, which are under consideration, the generalization of the conditions (a) and (b) shall be:

(aa) for each $y \in C(S;X)$ and $\psi \in C(S;X)$ the functions $t \to F(t) \times \times (t)(\psi(t), y(t))$ and $t \to G(t)(y(t), \psi(t))$ are determined $\forall\, t \in S$ and belong to $C(S;X^*)$, $C(S;Z)$ accordingly.

(bb) for $y \in C(S;X)$, $\psi \in C(S;Z)$ the operators $G(y,\cdot) \in (C(S;Z) \to \to C(S;Z))$, $F(\psi,\cdot) \in (C(S;X) \to C(S;X^*))$ and equally of Lipshitz, that is there are the constant $r_1$ and $r_2$, that the conditions come true:

$$\|F(\psi, y_1) - F(\psi, y_2)\|_{C(S;X^*)} \le r_1 \|y_1 - y_2\|_{C(S;X)}$$

$$\forall\, \psi \in C(S;Z),\ \forall\, y_1, y_2 \in C(S;Z),$$

$$\|G(y, \psi_1) - G(y, \psi_2)\|_{C(S;Z)} \le r_2 \|\psi_1 - \psi_2\|_{C(S;Z)}$$

$$\forall\, y \in C(S;X),\ \forall\, \psi_1, \psi_2 \in C(S;Z)\,.$$

We give for example the known statement, which is used with the application of principle of fixed point to the tasks under consideration.

**Lemma 4** [5]. If the operators $G$ satisfy the conditions (42), then for any $\psi_1, \psi_2 \in C(S;Z)$ and $\forall\, t \in S$

$$\|G\psi_1 - G\psi_2\|_{C([0;t];Z)} \le r_2 \|\psi_1 - \psi_2\|_{C([0;t];Z)}, \quad r_2 = \text{const}\,.$$

Let us give the example for Volterra's operators, which satisfy (42).

1. Let $h \in C(S)$, $0 \le h(t) \le t$ for $\forall\, t \in S$ and $\{Q(t),\ t \in S\}$ be the family of operators from $X \to X$, which satisfies the conditions:

- for each $x \in X$ the function $t \to Q(t)x$ is determined for $t \in S$ and belongs to $C(S;X)$;

- the operators $\{Q(t)\} \in (X \to X)$ (equally-relatively $t \in S$) is Lipshitz's continuous, i.e. there is such a constant *r* (independent on *t*), so for any $x, y \in X$ Lipshitz condition comes true.

$$\|Q(t)x - Q(t)y\|_X \le r\|x - y\|_X . \tag{43}$$

If we put

$$(Qu)(t) = Q(t)u(h(t)), \; u \in C(S;X),$$

then $Q$ will be Lipshitz's-continuous operator of Volterra from $(C(S;X) \to \to C(S;X))$.

2. Let the operator of Volterra $V$ satisfy the conditions (42). Then for $u \in C(S;X)$ we have

$$(Qu)(t) = \int_0^t k(t,s)(Vu)(s)ds, \quad k \in C^1(S \times S), \; t \in S;$$

the operator $Q$ will also be an operator of Volterra, which satisfies (8).

3. Let $Q$, $V$ be operators of Volterra, which satisfy (8). Then their linear combinations and composition $Q \circ V$ have the same properties.

The following theorem is true.

**Theorem 3**. Let $F$, $G$ be operators of Volterra, which satisfy the conditions (aa), (bb), and the family of linear operators $B(t)$ satisfy the inequality $\|B(t)\| < \frac{1}{r_1}$ $\forall t \in S$, $r_1$ — Lipshitz's constant from (bb).

Then the task

$$y(t) + B(t)(F(\psi, y))(t) = \varphi(t) \quad \forall t \in S, \tag{44}$$

$$\psi'(t) + (G(y,\psi))(t) = f(t), \tag{45}$$

$$\psi(0) = \gamma, \quad \psi \in C^1(S;Z) \tag{46}$$

has the solution $(y,\psi) \in C(S;X) \times C^1(S;Z)$ for any $\varphi \in (S;X)$, $f \in C(S;Z)$ and $\gamma \in Z$.

**Proof**. Due to the fact that in this case the principle of fixed point is used (but in other Banach spaces), the proof of the theorem is conducted analogically to the proof of the theorem 1. Integrating equations (45) in $[0;t]$ and considering the initial conditions we obtain:

$$\psi(t) = \gamma(\omega) - \int_0^t [(G(y,\psi))(s) - f(s)]ds,$$

$$t \in S, \; y \in C(S;X), \; \psi \in C(S;Z).$$

Here the integral is considered in the sense of Bohner.

We introduce the operator $U$, which follows the rule

$$(U\psi)(t) = \gamma - \int_0^t \left[ (G(y,\psi))(s) - f(S) \right] ds, \qquad (47)$$

where the function $f \in C(S;Z)$ and the operator $G \in (C(S;Z) \to C(S;Z))$. Proceeding from theorem on differentiality of nondesignated integral of Bohner (see [5], theorem 1.9, ch. 4), we obtain $U \in \left( C(S;Z) \to C^1(S;Z) \right)$.

We mark in equation (44)

$$(U_0 y)(t) = \varphi(t) - B(t)(F(\psi, y))(t), \ t \in S. \qquad (48)$$

We show that $U$, as the reflection of space $C(S;Z)$ in itself, with some $k \ge 0$, is compressed in $(C,k)$-norm, which is determined in lemma 2.

By the virtue of condition (bb) from (47), it follows that for any $\psi_1, \psi_2 \in C(S;Z)$

$$\|U\psi_1 - U\psi_2\|_{C(S;Z)} \le \int_0^t \|G(y,\psi_1) - G(y,\psi_2)\|_{C(S;Z)} ds =$$

$$= r_2 \int_0^t \sup_{0 \le \tau \le s} \left\{ \|\psi_1(\tau) - \psi_2(\tau)\|_Z e^{-ks} \right\} e^{ks} ds \le$$

$$\le r_2 \int_0^t \sup_{0 \le \tau \le s} \left\{ \|\psi_1 - \psi_2\|_{C([0;s];Z)} e^{-ks} \right\} e^{ks} ds =$$

$$= r_2 \|\psi_1 - \psi_2\|_{(C,k)} \int_0^t e^{ks} ds \le r_2 \|\psi_1 - \psi_2\|_{(C,k)} \frac{1}{k} (e^{kt} - 1).$$

Having multiplied both parts of this inequality by $e^{-kt}$, we obtain

$$\|U\psi_1 - U\psi_2\|_{C(S;Z)} e^{-kt} \le \frac{r_2}{k}(1 - e^{-kt}) \|\psi_1 - \psi_2\|_{(C,k)} \le \frac{r_2}{k}(1 - e^{-kT}) \|\psi_1 - \psi_2\|_{C,k}.$$

Considering in the left part the upper margine as for $t \in S$, we obtain

$$\|U\psi_1 - U\psi_2\|_{(C,k)} \le \frac{r_2}{k}(1 - e^{-kT}) \|\psi_1 - \psi_2\|_{(C,k)}.$$

If we choose $k \ge r_2$, then the reflection $U$ in $(C,k)$-norm will be compressed, i.e. there is an element $\psi_0 \in C(S;Z)$, which is a fixed point for this reflection:

$$\Psi_0 = U\psi_0.$$

Considering (47), we obtain

$$\psi_0(t) = \gamma - \int_0^t \left[ (G(y,\psi_0))(s) - f(s) \right] ds \quad \forall\, t \in S. \qquad (49)$$

Due to the fact that the right part of this equation has the continuous derivative as for $t$, then $\psi_0 \in C^1(S;Z)$ and $\psi_0'(t) + (G(y,\psi_0))(t) = f(t)$ $\forall t \in S$, $\psi_0(0) = \gamma$.

We consider the conditions under which the operator $U_0$ will be compressed.

According to (bb) from (48), it follows that $\forall\, y_1, y_2 \in C(S;X)$

$$\|U_0 y_1(t) - U_0 y_2(t)\|_X = \|B(t)(F(\psi, y_1))(t) - B(t)(F(\psi, y_2))(t)\|_X =$$

$$= \|B(t)[F(\psi, y_1)(t) - F(\psi, y_2)(t)]\|_X \le \|B(t)\| \cdot \|(F(\psi, y_1))(t) - (F(\psi, y_2))(t)\|_{X^*} \le$$

$$\le \|B(t)\| r_1 \|y_1 - y_2\|_{C(S;X)} \quad \forall t \in S.$$

Following the condition of the theorem $\|B(t)\| < \frac{1}{r_1}$, the operator $U_0$ is compressed.

In the system of the equation

$$y = \varphi(t) - B(t)(F(\psi, y))(t), \tag{50}$$

$$\psi(t) = \gamma - \int_0^t [(G(y,\psi))(s) - f(s)]ds, \tag{51}$$

giving in the right part of (50) the values $y = y_1, \psi = \psi_1$, where $(y_1, \psi_1)$ is some pair, we get $y_2(t) = \varphi(t) - B(t)(F(\psi_1, y_1))(t)$, from which $\psi_2(t) = \gamma - \int_0^t [(G(y_2, \psi_1))(s) - f(s)]ds$.

Analogically, substituting the obtained pair $(y_2, \psi_2)$, we find $y_3$. Then, substituting the pair $(y_3, \psi_2)$ in (51), we find $\psi_3$. Repeating such procedure, we find

$$y_{n+1} = \varphi(t) - B(t)(F(\psi_n, y_n))(t) \tag{52}$$

and

$$\psi_{n+1}(t) = \gamma - \int_0^t [(G(y_{n+1}, \psi_n))(s) - f(s)]ds. \tag{53}$$

Then, like in theorem 1, we have proved that the sequences $\{y_n\}$, $\{\psi_n\}$ are fundamental and, as a result, proceed to the fixed point $\{y_0, \psi_0\}$, which is the solution for the system of equation under consideration.

**CONCLUSIONS**

In this paper we have shown some results in the context of the mathematical models of mixed systems. They are the necessary base when setting and solving tasks for the optima control and the evaluation of parameters of objects states, which are described trough nonlinear initial-boundary tasks for integral and differential equations and their systems with partial derivatives, developing methods and algorithms of regularization of optimization tasks, composing terminal-measurable approximations and averaging-out schemes, synthesis of applied control systems for different processes etc.

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*Received 28.03.2006*

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