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METHOD OF APPROXIMATION OF EVOLUTIONARY INCLUSIONS AND VARIATIONAL INEQUALITIES BY STATIONARY

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The method of finite-difference approximations, advanced by C. Bardos and H. Brezis for the nonlinear evolutionary equations, is generalized on differential-operational inclusions which are tightly connected to evolutionary variational inequalities in Banach spaces.

INTRODUCTION

At studying of nonlinear evolutionary equations the some spread methods are used: Faedo-Galerkin, singular perturbations, difference approximations, nonlinear semigroups of operators and others [1, 2]. The dissemination of these approaches on evolutionary inclusions and variational inequalities encounters a series of basic difficulties. The method of nonlinear semigroups of operators in Banach spaces was developed for evolutionary inclusions in works of A.A. Tolstonogov [3], A.A. Tolstonogov and J.I. Umanskij [4], V. Barbu [2] and others. A method of singular perturbations H. Brezis [5] and Yu. Dubinskiy [6] on evolutionary inclusions have disseminated in A.N. Vakulenko's and V.S. Mel'nik works [7−9], a method of Galerkin's approximations in P.O. Kasyanov's works [10, 11].

In the present work the attempt to disseminate a method of difference approximations [1] on evolutionary inclusions and variational inequalities is undertaken for the first time.

PROBLEM FORMALIZATION

Let Φ be separable locally convex linear topological space; Φ' be the space identified to topologically conjugate to Φ space such, that $\Phi \subset \Phi'$; (f,φ) is the inner product (canonical pairing) of devices $f \in \Phi'$ and $\varphi \in \Phi$.

Let the three spaces V, H and V' are given, moreover

$$\Phi \subset V \subset \Phi', \quad \Phi \subset H \subset \Phi', \quad \Phi \subset V' \subset \Phi' \tag{1}$$

with continuous and dense embedding;

H is a Hilbert space (with inner product $(h_1,h_2)_H$ and corresponding norm $\|h\|_H$);
V be reflexive separable Banach space with norm $\|v\|_V$;
V' is the conjugate to V space with dual norm $\|f\|_{V'}$.

If $\varphi,\psi\in\Phi$, that $(\varphi,\psi)=(\varphi,\psi)_H$ is inner product of devices $\varphi\in V$ and $\psi\in V'$.

Let $V=V_1\bigcap V_2$ and $\|\cdot\|_V=\|\cdot\|_{V_1'}+\|\cdot\|_{V_2'}$, where $(V_i,\|\cdot\|_{V_i})$, $i=\overline{1,2}$ is reflexive separable Banach spaces, embedding $\Phi\subset V_i\subset\Phi'$ and $\Phi\subset V_i'\subset\Phi'$ is dense and continuous. Spaces $(V_i',\|\cdot\|_{V_i'})$, $i=\overline{1,2}$ are topologically conjugate to $(V_i,\|\cdot\|_{V_i})$ concerning the bilinear form $(\cdot,\cdot)$. Then $V'=V_1'+V_2'$.

Let $A:V_1\to V_1'$, $\varphi:V_2\to R$ be a functional, Λ is non-bounded operator, which operates from V to V' with definitional domain $D(\Lambda;V,V')$. The following problem on searching of solutions by a method of finite differences is considered (see [1, chapter 2.7]):

$$u\in D(\Lambda;V,V'), \tag{2}$$

$$\Lambda u+A(u)+\partial\varphi(u)\ni f, \tag{3}$$

where $f\in V'$ fixed element; $\partial\varphi:V_2\rightrightarrows V_2'$ is subdifferential from the functional φ (see [13]).

THE BASIC GUESSES

Let us assume, that a set Φ is dense in space

$$(V\bigcap V',\|v\|_V+\|v\|_{V'}). \tag{4}$$

Remark 1. From (4) it follows, that

$$V\bigcap V'\subset H. \tag{5}$$

Really, if $v\in\Phi$, that $\|v\|_H^2\le\|v\|_{V'}\|v\|_V$ whence, due to (4) it follows (5).

Remark 2. If $V\subset H$, it is possible to not introduce Φ and identifying H and H', at once receive the following line-up of embeddings:

$$V\subset H\subset V'. \tag{6}$$

Definition 1. The family of maps $\{G(s)\}_{s\ge0}$ refers to as a *continuous semigroup* in a Banach space X, if $\forall s\ge0$ $G(s)\in L(X;X)$, $G(0)=Id$, $G(s+t)=G(s)\circ G(t)$ $\forall s,t\ge0$, $G(t)x\xrightarrow{w}x$ as $t\to0+$ $\forall x\in X$.

Operator Λ. Let the family of maps $\{G(s)\}_{s\ge0}$ be such that $\{G(s)\}_{s\ge0}$ is continuous semigroup on V, H, V', that is there are three semigroups, defined in spaces V,H, and V' correspondingly, which coincide on Φ. Each of them we shall designate as $\{G(s)\}_{s\ge0}$;

$\{G(s)\}_{s\geq 0}$ is non-expanding semigroup in H,

$$\text{that is } \|G(s)\|_{L(H;H)} \leq 1 \quad \forall s \geq 0. \tag{7}$$

Further let $-\Lambda$ be the infinitesimal generator of a semigroup $\{G(s)\}_{s\geq 0}$ with a definitional domain $D(\Lambda;V)$ (accordingly $D(\Lambda;H)$ or $D(\Lambda;V')$) in V (accordingly in H or in V'). In virtue of [14, theorem 13.35] such generator exists, moreover, it is densely defined closed linear operator in space V (accordingly in H or in V').

Let $\{G^*(s)\}_{s\geq 0}$ be the semigroup conjugated to $G(s)$, which operates accordingly in V, H, and V'. Let $-\Lambda^*$ is the infinitesimal generator of a semigroup $\{G^*(s)\}_{s\geq 0}$ with definitional domain $D(\Lambda^*;V)$ in V, $D(\Lambda^*;H)$ in H and $D(\Lambda^*;V')$ in V'. The operator Λ^* in H (accordingly in V or in V') is conjugated in sense of the theory of unlimited operators to the operator Λ in H (accordingly in V or in V'). It takes place the following.

Lemma 1. The sets $D(\Lambda;V')\bigcap V$ and $D(\Lambda^*;V')\bigcap V$ are dense in V.

Proof. Really, $\forall u \in V \quad \forall \varepsilon > 0 \quad \exists \varphi \in \Phi: \quad \|u-\varphi\|_V < \varepsilon, \quad \varphi_n :=$ $= \left(I - \frac{1}{n}\Lambda\right)^{-1} \varphi \in D(\Lambda;V') \bigcap V$, $\varphi_n \to \varphi$ in V as $n \to \infty$.

The lemma is proved.

Now we define Λ as non-bounded operator, which operates from V to V' with definitional domain $D(\Lambda;V,V')$. Let us put

$$D(\Lambda;V,V') = \{v \in V \mid \text{the form} \quad w \to (v, \Lambda^* w) \text{ is continuous on}$$
$$D(\Lambda^*;V')\bigcap V \quad \text{in topology, induced from space } V\}. \tag{8}$$

Then there is unique element $\xi_v \in V' : (v, \Lambda^* w) = (\xi_v, w)$. If $v \in D(\Lambda;V')\bigcap$ $\bigcap V$, that $\xi_v = \Lambda v$. Thus, generally we can put $\xi_v = \Lambda v$, whence

$$(v, \Lambda^* w) = (\Lambda v, w) \quad \forall w \in D(\Lambda^*;V') \bigcap V. \tag{9}$$

If we enter on $D(\Lambda;V,V')$ the norm $\|v\|_V + \|\Lambda v\|_{V'}$, we receive a Banach space. Let us similarly define space $D(\Lambda^*;V,V')$.

Remark 3. If $V \subset H$, then

$$D(\Lambda;V,V') = V \bigcap D(\Lambda;V') \quad \text{and} \quad D(\Lambda^*;V,V') = V \bigcap D(\Lambda^*;V').$$

In case when V does not include in H we assume that

$$V \bigcap D(\Lambda;V') \text{ dense in } D(\Lambda;V,V'),$$
$$V \bigcap D(\Lambda^*;V') \text{ dense in } D(\Lambda^*;V,V'). \tag{10}$$

Remark 4. ([1, chapter 2, remark 7.5., 7.6.]).

$$(\Lambda v, v) \geq 0 \quad \forall v \in D(\Lambda; V, V'), \quad (\Lambda^* v, v) \geq 0 \quad \forall v \in D(\Lambda^*; V, V'). \qquad (11)$$

Let us enter some new denotations. Let Y be some reflexive Banach space. As $C_v(Y)$ we designate the system of all nonempty convex closed bounded subsets from Y. For nonempty subset $B \subset Y$ we consider the closed convex hull of the given set $\overline{\text{co}}(B) := \text{cl}_Y(\text{co}(B))$. With multi-valued map A it is comparable *upper* $[A(y), \omega]_+ = \sup_{d \in A(y)} < d, w >_Y$ and *lower* $[A(y), \omega]_- = \inf_{d \in A(y)} < d, w >_Y$ *function of support,* where $y, \omega \in Y$. Properties of the given maps are considered in works [15–17]. Later on $y_n \xrightarrow{w} y$ in Y will mean, that y_n weakly converges to y in space Y.

THE CLASSES OF MAPS

Let us consider the next classes of maps of pseudomonotone type:

Definition 2. Operator $A : V \to V'$ refers to *pseudomonotone*, if from $\{y_n\}_{n \geq 0} \subset V$, $y_n \xrightarrow{w} y_0$ in V, and $\overline{\lim_{n \to \infty}} (A(y_n), y_n - y_0) \leq 0$ it follows, that $\exists \{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1}$:

$$\underline{\lim_{k \to \infty}} (A(y_{n_k}), y_{n_k} - w) \geq (A(y_0), y_0 - w) \quad \forall w \in V.$$

Definition 3. The next set:

$$\partial \varphi(v) = \{ p \in V' \,|\, < p, u - v > \leq \varphi(u) - \varphi(v) \quad \forall\, u \in V \}$$

refers to *subdifferential map* form functional $\varphi : V \to \mathbf{R}$ in point $v \in V$.

Definition 4. Multi-valued map $A : V \rightrightarrows V^*$ refers to:

1) λ*-pseudomonotone*, if from $\{y_n\}_{n \geq 0} \subset V$, $y_n \xrightarrow{w} y_0$ in V and $\overline{\lim_{n \to \infty}} (d_n, y_n - y_0) \leq 0$, where $d_n \in \overline{\text{co}}\, A(y_n)$ $\forall n \geq 1$ it follows, that it is possible to choose such $\{y_{n_k}\}_{k \geq 0} \subset \{y_n\}_{n \geq 0}$, $\{d_{n_k}\}_{k \geq 0} \subset \{d_n\}_{n \geq 0}$ that

$$\forall\, w \in V \qquad \underline{\lim_{k \to \infty}} (d_{n_k}, y_{n_k} - w) \geq [A(y_0), y_0 - w]_-;$$

2) *bounded*, if A translates arbitrary bounded in V set in bounded in V^*;

3) *coercive*, if $\| v \|_V^{-1} [A(v), v]_+ \to +\infty$ as $\| v \|_V \to +\infty$;

4) satisfies *condition* (κ) if the map $V \ni v \to \| v \|_V^{-1} [A(v), v]_+ \in \mathrm{R}$ is bounded from below on bounded in $V \setminus \bar{0}$ sets, that is

$$\forall\, D \subset V \setminus \{\bar{0}\} - \text{bounded in } V \quad \exists\, c_1 \in \mathrm{R} : \ \frac{[A(v), v]_+}{\| v \|_V} \geq c_1 \quad \forall\, v \in D.$$

Remark, that the bounded multi-valued maps and monotone multi-valued operators, including subdifferential maps, are satisfying condition (κ).

Definition 5. Multivalued map $A: V \to C_v(V^*)$ satisfies *property* (M), if from $\{y_n\}_{n\geq 0} \subset V$, $d_n \in A(y_n)$ $\forall n \geq 1$: $y_n \xrightarrow{w} y_0$ in V, $d_n \xrightarrow{w} d_0$ in V', $\overline{\lim_{n\to\infty}} (d_n, y_n) \leq (d_0, y_0)$ it follows, that $d_0 \in A(y_0)$.

Definition 6. Operator $L: D(L) \subset V \to V^*$ refers to *maximally monotone*, if it is monotone and from $(w - L(u), v - u) \geq 0$ $\forall u \in D(L)$ it follows, that $v \in D(L)$ and $L(v) = w$.

Lemma 2. Let V, W be Banach spaces, densely and continuously embedded in locally convex linear topological space Y, $A: V \rightrightarrows V'$, $B: W \rightrightarrows W'$ — multi-valued λ-pseudomonotone maps and one of them is bound-valued. Then the multi-valued operator $A := A + B: V \bigcap W \rightrightarrows V' + W'$ is λ-pseudomonotone.

Proof. Let $y_n \xrightarrow{w} y$ in $X := V \bigcap W$ (that is $y_n \xrightarrow{w} y$ in V and $y_n \xrightarrow{w} y$ in W) and the next inequality is holds:

$$\overline{\lim_{n\to\infty}} < d_n, y_n - y >_X \leq 0, \tag{12}$$

where

$$d_n \in \overline{\text{co}}\, A(y_n) = \overline{\text{co}}\, A(y_n) + \overline{\text{co}}\, B(y_n). \tag{13}$$

Let us prove the last equality. It is obvious, that $\text{co}\, A(y_n) = \text{co}\, A(y_n) + \text{co}\, B(y_n)$ and, moreover, $\overline{\text{co}}\, A(y_n) \supset \overline{\text{co}}\, A(y_n) + \overline{\text{co}}\, B(y_n)$. Let us prove the inverse inclusion. Let x is a frontier point of $A(y_n)$. Then $\exists \{x_m\}_{m\geq 1} \subset \text{co}\, A(y_n) = \text{co}\, A(y_n) + \text{co}\, B(y_n)$: $x_m \xrightarrow{w} x$ in X as $m \to \infty$, because of Mazur theorem (see [14]), for an arbitrary convex set its weak and the strong closure is coincide. Hence, $\forall m \geq 1$ $\exists v_m \in A(y_n)$, $\exists w_m \in B(y_n)$: $v_m + w_m = x_m$ and, taking into account bound-valuededness of one of the maps and Banach-Alaoglu theorem, we obtain, within to a subsequence, $v_m \xrightarrow{w} v$ in V, $w_m \xrightarrow{w} w$ in W for some $v \in \overline{\text{co}}\, A(y_n)$, $w \in \overline{\text{co}}\, B(y_n)$. The statement (13) is proved. Consequently $d_n = d'_n + d''_n$, where $d'_n \in \overline{\text{co}}\, A(y_n)$, $d''_n \in \overline{\text{co}}\, B(y_n)$. From here, within to a subsequence, we obtain one of two inequalities:

$$\overline{\lim_{n\to\infty}} < d'_n, y_n - y >_V \leq 0, \qquad \overline{\lim_{n\to\infty}} < d''_n, y_n - y >_W \leq 0. \tag{14}$$

Without loss of generality, let us consider, that (within to a subsequence) $\overline{\lim_{n\to\infty}} < d'_n, y_n - y >_V \leq 0$. Then, due to λ-pseudomonotony of A, $\exists \{y_m\}_m \subset \{y_n\}_{n\geq 1}$:

$$\underline{\lim_{m\to\infty}} < d'_m, y_m - v >_V \geq [A(y), y - v]_- \quad \forall v \in V.$$

Let us put in last equality $v = y$, then

$$\underline{\lim}_{m\to\infty} < d'_m, y_m - y >_V \geq [A(y), y-y]_- = 0.$$

Hence, $\exists \lim_{m\to\infty} < d'_m, y_m - y >_V = 0$. Then, due to (12), $\overline{\lim}_{n\to\infty} < d'_m, y_m - y >_W \leq 0$. Taking into account (14), λ-pseudomonotony of A and B, we have

$$\underline{\lim}_{k\to\infty} < d'_{n_k}, y_{n_k} - v >_V \geq [A(y), y-v]_- \quad \forall v \in V,$$

$$\underline{\lim}_{k\to\infty} < d''_{n_k}, y_{n_k} - w >_W \geq [B(y), y-w]_- \quad \forall w \in W.$$

Then from last two relations it follows

$$\underline{\lim}_{k\to\infty} < d_{n_k}, y_{n_k} - x >_X \geq \underline{\lim}_{k\to\infty} < d'_{n_k}, y_{n_k} - x >_V + \underline{\lim}_{k\to\infty} < d''_{n_k}, y_{n_k} - x >_W \geq$$

$$\geq [A(y), y-x]_- + [B(y), y-x]_- = [A(y), y-x]_- \quad \forall x \in V \cap W.$$

The lemma is proved.

Lemma 3. Let V, W be Banach spaces, densely and continuously embedded in locally convex linear topological space Y, $A: V \rightrightarrows V'$, $B: W \rightrightarrows W'$ are multi-valued coercive maps, which satisfies condition (κ). Then the multi-valued operator $A := A + B : V \cap W \rightrightarrows V' + W'$ is coercive.

Proof. We obtain this statement arguing by contradiction. Let's assume, that $\exists \{x_n\}_{n\geq 1} : \| x_n \|_X = \| x_n \|_V + \| x_n \|_W \to +\infty$ as $n \to \infty$, but $\sup_{n\geq 1} \dfrac{[A(x_n), x_n]_+}{\| x_n \|_X} < +\infty$.

Case 1. $\| x_n \|_V \to +\infty$ as $n \to \infty$, $\| x_n \|_W \leq c \ \ \forall n \geq 1$;

$$\gamma_A(r) := \inf_{\|v\|_V = \gamma} \frac{[A(v), v]_+}{\| v \|_V}, \quad \gamma_B(r) := \inf_{\|w\|_W = \gamma} \frac{[B(w), w]_+}{\| w \|_W}, \quad r > 0.$$

Remark, that $\gamma_A(r) \to +\infty, \gamma_B(r) \to +\infty$ as $r \to +\infty$. Then $\forall n \geq 1$ $\| x_n \|_V^{-1} [A(x_n), x_n]_+ \geq \gamma_A(\| x_n \|_V) \| x_n \|_V$ and $\dfrac{[A(x_n), x_n]_+}{\| x_n \|_X} \geq \gamma_A(\| x_n \|_V) \times \dfrac{\| x_n \|_V}{\| x_n \|_X} \to +\infty$ as $\| x_n \|_V \to +\infty$ and $\| x_n \|_W \leq c.$

In this case, due to condition (κ), $\forall n \geq 1$

$$\frac{[B(x_n), x_n]_+}{\| x_n \|_X} \geq \gamma_B(\| x_n \|_W) \frac{\| x_n \|_W}{\| x_n \|_X} \geq c_1 \frac{\| x_n \|_W}{\| x_n \|_X} \to 0 \quad \text{at} \quad n \to \infty,$$

where $c_1 \in \mathrm{R}$ is the constant from condition (κ). It is clear, that

$$\frac{[A(x_n), x_n]_+}{\| x_n \|_X} = \frac{[A(x_n), x_n]_+}{\| x_n \|_X} + \frac{[B(x_n), x_n]_+}{\| x_n \|_X} \to +\infty \quad \text{as} \quad n \to \infty.$$

We have an inconsistency with boundedness of the left part of the given expression.

Case 2. The case $\| x_n \|_V \le c \quad \forall n \ge 1$ and $\| x_n \|_W \to \infty$ as $n \to \infty$ is investigated similarly.

Case 3. Let us consider the situation, when $\| x_n \|_V \to \infty$ and $\| x_n \|_W \to \infty$ as $n \to \infty$. Then,

$$+\infty > \sup_{n\ge 1} \frac{[A(x_n), x_n]_+}{\| x_n \|_X} \ge \gamma_A(\| x_n \|_V) \frac{\| x_n \|_V}{\| x_n \|_V + \| x_n \|_W} +$$

$$+ \gamma_B(\| x_n \|_W) \frac{\| x_n \|_W}{\| x_n \|_V + \| x_n \|_W} . \qquad (15)$$

It is obvious, that $\forall n \ge 1$ $\frac{\| x_n \|_V}{\| x_n \|_X} > 0$ and $\frac{\| x_n \|_W}{\| x_n \|_X} > 0$. And, if even one of limits, for example $\frac{\| x_n \|_V}{\| x_n \|_X} \to 0$, that $\frac{\| x_n \|_W}{\| x_n \|_X} = 1 - \frac{\| x_n \|_V}{\| x_n \|_X} \to 1$. We have an inconsistency with (15).

The lemma is proved.

THE MAIN RESULT

Theorem. Let a) $A : V_1 \to V_1'$ be bounded pseudomonotone on V_1 operator, which satisfies the following coercive condition:

$$\frac{(A(u), u)}{\| u \|_{V_1}} \to +\infty \quad \text{as} \quad \| u \|_{V_1} \to +\infty ; \qquad (16)$$

b) functional $\varphi : V_2 \to \mathrm{R}$ is convex, lower semicontinuous and the following takes place:

$$\frac{\varphi(v)}{\| v \|_{V_2}} \to +\infty \quad \text{as} \quad \| v \|_{V_2} \to +\infty ; \qquad (17)$$

c) The operator Λ satisfies all listed above conditions, including conditions (7) and (10).

Then for every $f \in V'$ there exists such u, that satisfies (2) and (3).

Remark 5. If $V \subset H$, inclusion (2) implies, that $u \in V \cap D(\Lambda; V')$.

Proof. *The approximate solutions*. Natural approximation of inclusion (3) is inclusion

$$\frac{I - G(h)}{h} u_h + A(u_h) + \partial\varphi(u_h) \ni f \quad (h > 0) . \qquad (18)$$

Though, if V does not include in H (18), generally speaking, has no solutions, and it is necessary to modify the given inclusion in appropriate way. We choose such sequence $\theta_h \in (0,1)$, that

$$\frac{1 - \theta_h}{h} \to 0 \quad \text{as} \quad h \to 0 . \qquad (19)$$

Let us put $\theta_h = 1$ when $V \subset H$. Further, we take

$$\Lambda_h = \frac{I - \theta_h G(h)}{h} \tag{20}$$

and also replace (18) with the inclusion

$$\Lambda_h u_h + A(u_h) + \partial\varphi(u_h) \ni f. \tag{21}$$

Lemma 4. Inclusion (21) has a solution $u_h \in V \bigcap H$.

Proof. Let us enter the map

$$B = \Lambda_h + A : H \bigcap V_1 \to H + V_1'. \tag{22}$$

We consider the following variation inequality:

$$(B(u_h), v - u_h) + \varphi(v) - \varphi(u_h) \ge (f, v - u_h) \quad \forall v \in V \bigcap H. \tag{23}$$

Let us prove the existence of such $u_h \in V \bigcap H$, that is a solution of the given inequality. The given statement follows from [15, theorem 7], if to put $V = H \bigcap V_1$, $W = V_2$, $A = B$, $\varphi = \varphi$ and under condition of realization

Lemma 5. Operator B satisfies to the following conditions:

$$\text{i)}\ \frac{(B(u), u)}{\| u \|_{H \cap V_1}} \to +\infty \quad \text{as} \quad \| u \|_{H \cap V_1} \to \infty; \tag{24}$$

$$\text{ii)}\ B \ \ is \ \ \text{pseudomonotone} \ \ \text{on} \ \ H \bigcap V_1; \tag{25}$$

$$\text{iii)}\ B \ \ is \ \ \text{bounded} \ \ \text{on} \ \ H \bigcap V_1. \tag{26}$$

Proof. i) As $G(s)$ is non-stretched on H, then $\forall v \in H$

$$(\Lambda_h v, v) = \frac{1}{h}(v - \theta_h G(h) v, v) \ge \frac{1}{h}\left(\| v \|_H^2 - \theta_h \| G(s) v \|_H \| v \|_H\right) \ge$$

$$\ge \frac{1 - \theta_h}{h} \| v \|_H^2. \tag{27}$$

From here it follows the coercive condition and condition (κ) for Λ_h on H. Thus, due to (2), we can use lemma 3 for maps $A = \Lambda_h$ on $V = H$ and $B = A$ on $W = V_1$, whence it follows (24), if we prove, that A satisfies condition (κ). Really, if it is not true, then $\exists \{w_n\}_{n \ge 1} \subset V_1 \setminus \overline{0}$ such bounded in W, that $\| w_n \|_{V_1}^{-1} [A(w_n), w_n]_+ \to -\infty$ as $n \to \infty$, but in virtue of boundedness of A, we have

$$\| w_n \|_{V_1}^{-1} [A(w_n), w_n]_+ = \| w_n \|_{V_1}^{-1} (A(w_n), w_n) \ge -\sup_{n \ge 1} \| A(w_n) \|_{V_1} > -\infty.$$

iii) The boundedness of B on $H \bigcap V_1$ follows from the boundedness of Λ_h on H and A on V_1. The boundedness of Λ_h on H immediately follows from the definition of Λ_h and estimation (6).

ii). Let us prove the pseudomonotony of B on $H\bigcap V_1$. For this purpose we use lemma 2 with $A=\Lambda_h$ on $V=H$ and $B=A$ on $W=V_1$. From here, due to the pseudomonotony and to the property of bound-valuedness of A on V_1, it is enough to prove pseudomonotony of Λ_h on H. Let

$$y_n \to y \quad \text{in} \quad H, \quad \overline{\lim_{n\to\infty}}(\Lambda_h y_n, y_n - y) \le 0.$$

Then, from estimation (27) we have

$$\underline{\lim_{n\to\infty}}(\Lambda_h y_n, y_n - y) \ge \underline{\lim_{n\to\infty}}(\Lambda_h y_n - \Lambda_h y, y_n - y) + \underline{\lim_{n\to\infty}}(\Lambda_h y, y_n - y) \ge 0 + 0 = 0 .$$

Hence $\exists \lim\limits_{n\to\infty}(\Lambda_h y_n, y_n - y) = 0$. Further, $\forall u \in H$, $\forall s > 0$ let $w := y +$ $+ s(u-y)$. Then

$$s(\Lambda_h y_n, y-u) \ge -(\Lambda_h y_n, y_n - y) + (\Lambda_h w, y_n - y) - s(\Lambda_h w, u-y) \quad \forall n \ge 1$$

and

$$s\,\underline{\lim_{n\to\infty}}(\Lambda_h y_n, y-u) \ge -s(\Lambda_h w, u-y) \Leftrightarrow \underline{\lim_{n\to\infty}}(\Lambda_h y_n, y-u) \ge -(\Lambda_h w, u-y) .$$

Let $s \to 0+$ then $\underline{\lim\limits_{n\to\infty}}(\Lambda_h y_n, y-u) \ge -(\Lambda_h y, u-y) = (\Lambda_h y, y-u)$ and

$$\underline{\lim_{n\to\infty}}(\Lambda_h y_n, y_h - u) \ge \underline{\lim_{n\to\infty}}(\Lambda_h y_n, y_h - y) +$$

$$+ \underline{\lim_{n\to\infty}}(\Lambda_h y_n, y-u) \ge (\Lambda_h y, y-u) \quad \forall u \in H .$$

Thus we have the required statement.

The lemma is proved.

To complete the proof of lemma 4 it is necessary to show, that for fixed $u_h \in H\bigcap V_1$ the variation inequality (23) is equivalent to inclusion (22). If $v \in H\bigcap V_1$ is arbitrary, then, by definition of subdifferential map, the inequality (23) is equivalent to $f - B(u_h) \in \partial\varphi(u_h)$, that in turn, by definition of B, it is equivalent to (22).

The lemma is proved.

The boundary transition on h. From lemma 4 for every $h>0$ the existence of such $u_h \in H \cap V_1$ and $d_h \in \partial\varphi(u_h)$, that

$$\Lambda_h u_h + A(u_h) + d_h = f . \tag{28}$$

is follows. If we put in (23) $v=\bar{0}$, we obtain

$$(B(u_h), u_h) + \varphi(u_h) \le (f, u_h) + \varphi(\bar{0}). \tag{29}$$

Let us prove boundedness of $\{u_h\}_{h>0}$ in V as h close to zero. For this purpose we use advantage coercive conditions (16) and (24). Let us assume, that $\| u_h \|_V = \| u_h \|_{V_1} + \| u_h \|_{V_2} \to \infty$.

Case 1. $\| u_h \|_{V_1} \to \infty$, $\| u_h \|_{V_2} \le c$;

$$\gamma_B(r) := \inf_{\|u\|_{V_1}=r} \frac{(B(u),u)}{\|u\|_{V_1}}, \quad \gamma_\varphi(r) := \inf_{\|u\|_{V_2}=r} \frac{\varphi(u)}{\|u\|_{V_2}}, \quad r > 0 .$$

Remark, that $\gamma_B(r) \to +\infty$ and $\gamma_\varphi(r) \to +\infty$ as $r \to +\infty$. Then $\|u_h\|_{V_1}^{-1} (B(u_h),u_h) \ge \gamma_B(\|u\|_{V_1})\|u\|_{V_1}$ and

$$\|f\|_{V'} \leftarrow \|f\|_{V'} + \frac{\varphi(\bar{0})}{\|u_h\|_V} \ge \frac{(f,u_h)+\varphi(\bar{0})}{\|u_h\|_V} \ge \frac{(B(u_h),u_h)+\varphi(u_h)}{\|u_h\|_V} \ge$$

$$\ge \frac{\gamma_B(\|u_h\|_{V_1})\|u_h\|_{V_1}}{\|u_h\|_V} + \frac{\gamma_\varphi(\|u_h\|_{V_2})\|u_h\|_{V_2}}{\|u_h\|_V} \ge$$

$$\ge \frac{\gamma_B(\|u_h\|_{V_1})\|u_h\|_{V_1}}{\|u_h\|_{V_1}+c} + \frac{\gamma_\varphi(\|u_h\|_{V_2})\|u_h\|_{V_2}}{\|u_h\|_V} \to +\infty \quad as \quad \|u_h\|_V \to \infty .$$

We have an inconsistency with boundedness of the left part of the given inequality. It is necessary to notice, that last item in a right-side of last inequality tends to zero. It follows from boundedness from below of φ on the bounded sets (see [13]).

Case 2. The case $\|u_h\|_{V_1} \le c$, $\|u_h\|_{V_2} \to \infty$ is investigated similarly.

Case 3. Let us consider the situation, when $\|u_h\|_{V_1} \to \infty$, $\|u_h\|_{V_2} \to \infty$. Then,

$$\|f\|_{V'} \leftarrow \|f\|_{V'} + \frac{\varphi(\bar{0})}{\|u_h\|_V} \ge \frac{\gamma_B(\|u_h\|_{V_1})\|u_h\|_{V_1}}{\|u_h\|_{V_1}+\|u_h\|_{V_2}} + \frac{\gamma_\varphi(\|u_h\|_{V_2})\|u_h\|_{V_2}}{\|u_h\|_{V_1}+\|u_h\|_{V_2}} . \quad (30)$$

It is obvious, that $\frac{\|u\|_{V_1}}{\|u\|_V} > 0$ and $\frac{\|u\|_{V_2}}{\|u\|_V} > 0$. And, if even one of boundaries, for example, $\frac{\|u\|_{V_1}}{\|u\|_V} \to 0$, that $\frac{\|u\|_{V_2}}{\|u\|_V} = 1 - \frac{\|u\|_{V_1}}{\|u\|_V} \to 1$. We have an inconsistency in (30). Thus,

$$u_h \quad \text{are bounded in} \quad V \quad \text{as} \quad h \to 0 . \quad (31)$$

Prove, that

$$d_h \quad \text{are bounded in} \quad V_2' \quad \text{as} \quad h \to 0 . \quad (32)$$

First, from equality (28) we receive:

$$\sup_n (d_{h_n}, u_{h_n}) < \infty \quad \forall \{h_n\} \subset (0,+\infty) : \ h_n \to 0 \quad \text{as} \quad n \to \infty . \quad (33)$$

Due to $u_h \in H$, from equality (28), estimation (31) and boundednesses of an operator A we have

$$\sup_n (d_{h_n}, u_{h_n}) = \sup_n (f, u_{h_n}) + \sup_n (-A(u_{h_n}), u_{h_n}) +$$

$$+ \sup_n (-\Lambda_{h_n} u_{h_n}, u_{h_n}) \le \|f\|_V' \sup_n \|u_{h_n}\|_V + \sup_n \|A(u_{h_n})\|_{V'} \sup_n \|u_{h_n}\|_V < +\infty .$$

Now, in virtue of (33), we prove (32). From $d_{h_n} \in \partial\varphi(y_{h_n})$ and from definition of subdifferential map, $\forall\, v \in V_2$

$$\sup_n (d_{h_n}, v) \le \sup_n (d_{h_n}, y_{h_n}) + \sup_n (d_{h_n}, v - y_{h_n}) \le \sup_n (d_{h_n}, y_{h_n}) + \varphi(v) - \varphi(y_{h_n}) \le$$

$$\le \sup_n (d_{h_n}, y_{h_n}) + \varphi(v) - \inf_n \varphi(y_{h_n}) < +\infty,$$

as functional φ is bounded from below on bounded sets. From here, under Banach-Steingauss theorem (32) is follows.

From (31) and boundedness of an operator A on V_1 it follows, that

$$A(u_h) \quad \text{are bounded in} \quad V_1' \quad \text{as} \quad h \to 0. \qquad (34)$$

From equality (28), estimates (31), (32) and (34), under Banach-Alaoglu theorem, the existence of such subsequences $\{u_{h_n}\}_{n\ge1} \subset \{u_h\}_{h>0}$, $\{d_{h_n}\}_{n\ge1} \subset \subset \{d_h\}_{h>0}$, $\{A(u_{h_n})\}_{n\ge1} \subset \{A(u_h)\}_{h>0}$ $(0 < h_n \to 0)$, which further we will designate simply as $\{u_h\}_{h>0}$, $\{d_h\}_{h>0}$, $\{A(u_h)\}_{h>0}$ accordingly, and elements $u \in V$, $\chi \in V_1$, $d \in V_2$ the next convergences

$$u_h \xrightarrow{w} u \quad \text{in} \quad V \quad A(u_h) \xrightarrow{w} \chi \quad \text{in} \quad V_1' \quad d_h \xrightarrow{w} d$$

$$\text{in} \quad V_2' \quad L_h u_h \xrightarrow{w} Lu \quad \text{in} \quad V' \qquad (35)$$

are follows, in particular,

$$v_h := A(u_h) + d_h \xrightarrow{w} \chi + d =: w \quad \text{in} \quad V'. \qquad (36)$$

Let us enter the following map: $C(v) = A(v) + \partial\varphi(v) : V \to C_v(V')$. Now prove, that the given map satisfies property (M). For this purpose it is enough to show λ-pseudomonotony of C on V. If C is λ-pseudomonotone on V and $\{y_n\}_{n\ge0} \subset V$, $d_n \in C(y_n)$ $\forall\, n \ge 1$:

$$y_n \xrightarrow{w} y_0 \quad \text{in} \quad V, \quad d_n \xrightarrow{w} d_0 \quad \text{in} \quad V' \quad \text{and} \quad \overline{\lim_{n\to\infty}} (d_n, y_n) \le (d_0, y_0),$$

then

$$\overline{\lim_{n\to\infty}} (d_n, y_n - y_0) \le \overline{\lim_{n\to\infty}} (d_n, y_n) + \overline{\lim_{n\to\infty}} (d_n, -y_0) \le (d_0, y_0) - (d_0, y_0) = 0.$$

Hence, due to λ-pseudomonotony of C it follows, that $\exists \{y_{n_k}\}_{k\ge1} \subset \subset \{y_n\}_{n\ge1}$, $\{d_{n_k}\}_{k\ge1} \subset \{d_n\}_{n\ge1}$:

$$\forall\, w \in V \quad \underline{\lim_{k\to\infty}} (d_{n_k}, y_{n_k} - w) \ge [C(y_0), y_0 - w]_-.$$

From here

$$[C(y_0),\, y_0 - w]_- \le \underline{\lim_{k\to\infty}} (d_{n_k},\, y_{n_k} - w) \le \overline{\lim_{n\to\infty}} (d_n, y_n - w) \le$$

$$\leq (d_0, y_0 - w) \quad \forall w \in V .$$

Hence $d_0 \in C(y_0)$. Thus C satisfies condition (M) on V.

In turn, lemma 2, pseudomonotony and bounded-valuedness of A on V_1 provides the last, if to prove λ-pseudomonotony of $\partial\varphi$ on V_2. As it is known, the last statement follows from [20.III, lemma 2, remark 2].

We use the fact, that C satisfies property (M) on V. Let us take v from $V \bigcap D(\Lambda^*; V')$. From (28) and (36) it follows, that

$$(u_h, \Lambda_h^* v) + (v_h, v) = (f, v) . \tag{37}$$

But

$$\Lambda_h^* v = \frac{I - G(h)^*}{h} v + \frac{I - \theta_h}{h} G(h)^* v \tag{38}$$

and due to (20), $\Lambda_h^* v \to \Lambda^* v$ in V'; and consequently, as h tends to zero in (37) we receive:

$$(u, \Lambda^* v) + (w, v) = (f, v) \quad \forall v \in \mathrm{V} \bigcap D(\Lambda^*; V')$$

and (in virtue of (7), (8)) $u \in D(\Lambda, V, V')$

$$\Lambda u + w = f$$

and we prove the theorem, if we show that

$$w \in C(u) . \tag{39}$$

On the other hand, because of (28) and (36) for $v \in \mathrm{V} \bigcap D(\Lambda;\, V') \subset \mathrm{H}$, we have

$$(v_h, u_h - v) = (f, u_h - v) - (\Lambda_h v, u_h - v) - (\Lambda_h (u_h - v), u_h - v) \leq$$

$$\leq (f, u_h - v) - (\Lambda_h v, u_h - v) ,$$

as $\Lambda_h \geq 0$ in $\Lambda(H; H)$. From here

$$\limsup (v_h, u_h) \leq (w, v) - (f, u - v) - (\Lambda v, u - v) \quad \forall v \in \mathrm{V} \bigcap D(\Lambda; V') .$$

But, due to (9), the same inequality is fulfilled $\forall v \in D(\Lambda; V, V')$, and when $v = u$ we obtain

$$\limsup (v_h, u_h) \leq (w, u) ,$$

and also (39), because of C is the operator of type (M). The theorem is proved.

Example. Let Ω in $\mathbf{R}^n$ be a bounded region with regular boundary $\partial\Omega$, $S = [0, T]$ be finite time interval, $Q = \Omega \times (0; T)$, $\Gamma_T = \partial\Omega \times (0; T)$. As operator A we take $(Au)(t) = A(u(t))$, where

$$A(\varphi) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial \varphi}{\partial x_i} \right|^{p-2} \frac{\partial \varphi}{\partial x_i} \right) + |\varphi|^{p-2} \varphi \tag{40}$$

(see [1, chapter 2.9.5]); V is closed subspace in Sobolev space $W^{1,p}(\Omega)$, $p>1$ such, that

$$W_0^{1,p}(\Omega)\subset V\subset W^{1,p}(\Omega) \qquad (41)$$

and

$$V_1=L_p(0,T;V),\quad H=L_2(0,T;L_2(\Omega)),\quad V_2=L_2(0,T;L_2(\Omega)).$$

We consider convex lower semicontinuous coercive functional $\psi:\mathbf{R}\to\mathbf{R}$ and its subdifferential $\Phi:\mathbf{R}\rightrightarrows\mathbf{R}$, that satisfies growth condition.

If we put $V=V_1\bigcap V_2$ (from here $V'=L_q(0,T;V^*)+L_2(0,T;L_2(\Omega))$, where $\frac{1}{p}+\frac{1}{q}=1$), we obtain the situation (6), if $p\geq 2$. At $1<p<2$ the common case takes place, if to take $\Phi=D(0,T;V)$ (see [1]).

As an operator Λ we take the derivation operator in sense of space of scalar distributions $D^*(0,T;V^*)$, $D(\Lambda;V,V'):=W=\{y\in V\bigcap H \mid y'\in H+V'\}$

$$G(s)\varphi(t):=\{\varphi(t-s)\ \text{ at }\ t\geq s;\ 0\ \text{ at }\ t\leq s\}.$$

Due to [1, chapter 2.9.5] and to the theorem, the next problem:

$$\frac{\partial y(x,t)}{\partial t}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial y(x,t)}{\partial x_i}\right|^{p-2}\frac{\partial y(x,t)}{\partial x_i}\right)+$$

$$+|y(x,t)|^{p-2}\,y(x,t)+\Phi(y(x,t))\ni f(x,t)\quad \text{a.e. on } Q, \qquad (42)$$

$$y(x,0)=0\quad \text{a.e. on } \Omega, \qquad (43)$$

$$\frac{\partial y(x,t)}{\partial \nu_A}=g(x,t)\quad \text{a.e. on } \Gamma_T, \qquad (44)$$

has a solution $y\in W$, obtained by finite differences method. Remark, that in (42)–(44): $f\in V'$, $y_0\in L_2(\Omega)$ are fixed elements.

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