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CELLULAR AUTOMATA MODELS WITH COMPLEX VALUED TRANSITION FUNCTIONS

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Abstract. The new class of mathematical models for computation theory is considered — namely cellular automata (CA) with branching complex-valued transition functions. The key point is possible multivaluedness of cell's states with such transition functions. Different cases with complex-value transition functions had been considered. Dynamics CA on one branch and on different isolated branches are described. Also the case of transitions of states between branches is proposed. The case of continuous-valued CA and their finite-valued approximations are discussed. The problem of approximation of multivalued CA is stated.

Keywords: cellular automata, complex-valued transition functions, Riemann surfaces, continuous-valued CA, branching, approximation of multivaluedness, computations.

INTRODUCTION

Typically, classical cellular automatic machines (CA) have a cell assembly structure with a certain set of possible states. The states change at discrete points in time according to certain rules. Up to now, as far as we know, these dynamic rules have been defined using unambiguous transition functions. Recently, there has been a need to investigate CA with multi-valued transition functions and multi-valued states of the cells [1, 2]. The multi-valued case is much more difficult then single-valued case. Further study of such new CA (with multivaluedness) depends on examining specific examples of such objects. In this paper, we propose complex-valued cellular automata with a branched (multi-valued) structure with values on the Riemann surface. CA with the location of cells on the Riemann surface is also considered. Using complex-valued CA allows using the rich set of mathematical tools of complex-valued analysis.

So we propose the setting of many research problems in case of complex-valued CA. The structure of the paper is the next. In the section 2 we give the formal description of common cellular automata.

The section 3 is devoted to remembering some elements of complex-valued analysis. One — dimensional case of CA is considered in section 4. There are proposed many new problems for investigations. One of the main problems is approximation problem. Such problem is very important for approximate calculation of multivalued solutions.

DESCRIPTION OF THE CELLULAR AUTOMATON

Here we give the formal description of cellular automata with anticipation. First of all we follow the papers on CA (see for example [1–7]). We pose here the description one-dimensional CA from [1] but the description can be modified to many-dimensional cases.

One-dimensional CA is represented by an array of cells x_i where $i \in \Sigma$ (integer set) and each x takes a value from a finite alphabet Σ. Thus, a sequence of cells $\{x_i\}$ of finite length n represents a string or global configuration c on Σ. This way, the set of finite configurations will be represented as Σ^n. An evolution is represented by a sequence of configurations $\{c^t\}$ given by the mapping $\Phi : \Sigma^n \to \Sigma^n$; thus their global relationship is as follows:

$$\Phi(c^t) \to c^{t+1}.$$

Where t time steps and every global state c of affairs are defined by a sequence of cell states. Also the cell states in configuration c^t are updated at the next configuration c^{t+1} simultaneously by a local function φ as follow s':

$$\varphi(x_{i-r}^t, \ldots, x_i^t, \ldots, x_{i+r}^t) \to x_i^{t+1};$$

φ — multi-valued complex function. Note that classical studies assume that an unambiguous transition function [3–6]. At the same time, there may be various variants of using branching complex-digit functions with Riemann surfaces. The simplest cases are given here, which convey some of the behaviour of this class of objects. Let us preliminarily recall the description of functions with Riemann surfaces.

RIEMANN SURFACES

Riemann's hierarchy is the traditional name for one-dimensional complex differentiable diversity in comprehensive analysis. Such surfaces were systematically studied by Bernhard Riemann (Fig. 1, 2) Examples of Riemann surfaces include the complex plane and sphere of Riemann. The surface of Riemann allows geometrically to present multi-valued functions of a complex variable in such a way that each of its points corresponds to one value of multi-valued function, and at continuous movement on a surface the function also changes continuously. The canonical view of the Riemann surface is the representation as a flatbread with some number of holes [8–11].

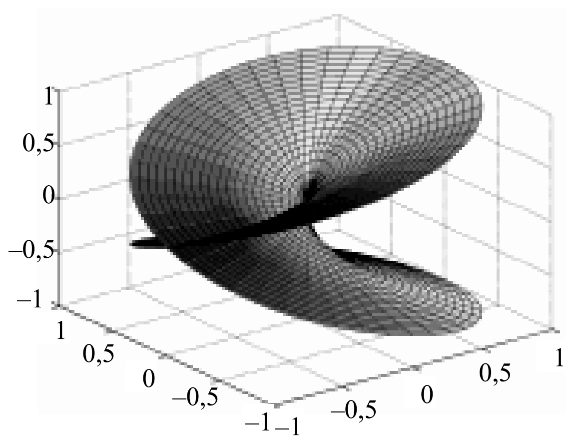

Fig. 1. Riemann surface for the function $f(z) = \sqrt{z}$

According to Felix Klein, the idea of the Riemann surface still belongs to Galois: in his suicide letter he mentions some research on “ambiguity of functions” among his achievements (Fig. 3) [11]. The topological characteristic of the Riemann surface is the genus; the genus surface is a sphere, the genus surface is a torus [8–11].

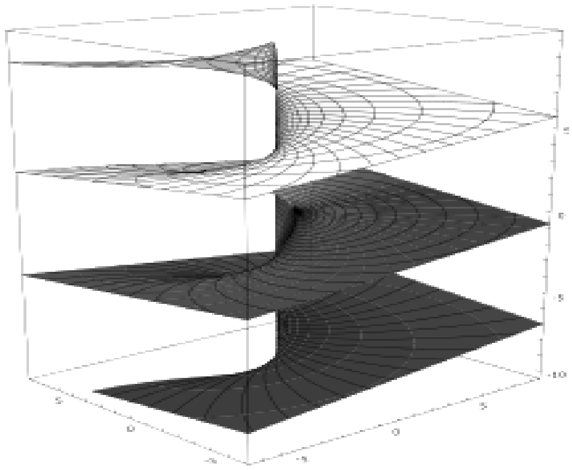

Fig. 2. Riemann surface for function $f(x) = \log z$

As an example, we also give the function of a complex variable with a single branching point, for example

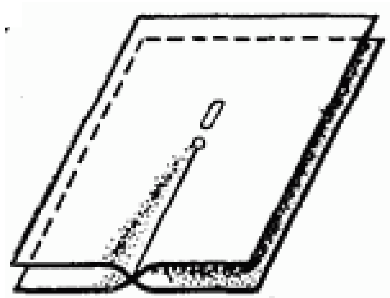

Fig. 3. Details of the construction of the Riemann surface function $f(z) = \sqrt{z}$

ONE-DIMENSIONAL CASE

In this paragraph we will illustrate the new research problems in the simplest cases.

One branching point. One branching point where branching conditions are not met for any point (e.g. all values during evolution are on the same branch of a function, e.g. on a function branch. Then the branching point is the only point, but let the CA values be outside the vicinity of the branching point. This corresponds to the class of unambiguous CA, but with certain generalizations. Namely, in our general case, CA looks like:

$$x_i^{n+1} = \varphi(\{\vec{x}^n\}_N; \vec{\alpha}),$$

x_i^n — state value of i cell at n moment of time ($n = 0,1,2,...$). In the case of classic CA, such as a game of life, the state of the cell is 0 или 1. In our case, the complex function φ, $x_i^n \in M$, $M \subset C^1$ where M — some subset of the space of complex numbers. Very conventionally, this can be represented in the form of a figure (Fig. 4)

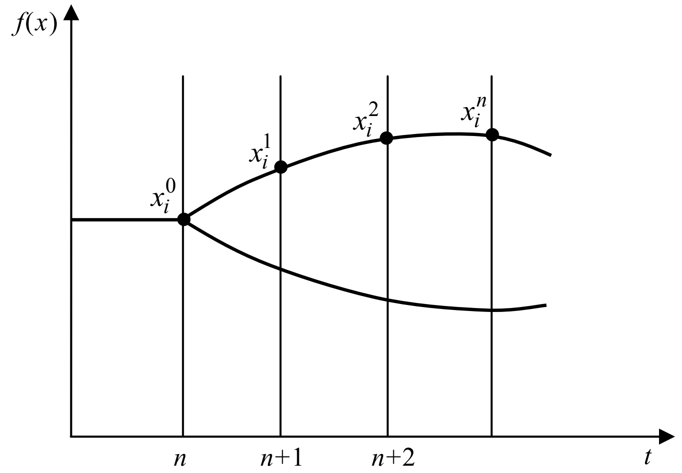

Fig. 4. Evolution of one cell by one branch

Here we represent the evolution of the values of one cell (Fig. 5). Other cells in case without branching have the same qualitative evolution depending on initial condition in the neighborhood of cell. Note that just in such case we have essential difference from the classical case. We can assume a whole set of values M, including $\text{card}\,M = \infty$, when a continuum of values is also allowed. (Remark that in classical case $M = \{0,1\}$. Note that in general case it is possible to consider approximations of a continuum set M using a discrete set $M_K \subset M$ with a finite number K of possible cell values. This can be roughly seen in the figure

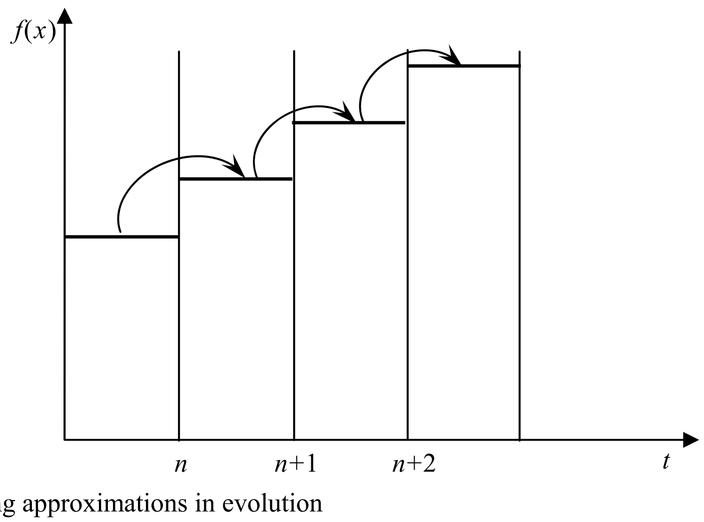

Fig. 5. Using approximations in evolution

An interesting task is the rigorous mathematical study of the applicability of such approximation, as well as the crystallization of the approximation.

An interesting task is the rigorous mathematical investigation of the applicability of such approximation as well as questions of marginal behaviour over time.

It can be assumed that assigning a rule to the CA transition as an analytical function can simplify the investigation of such questions.

Case of branching of transition function. A more complicated case is when the branching point of the transition function appears in the value range for only one cell. An illustration of chiefs appearing in the value range for only one cell is given in the figure below (Fig. 6).

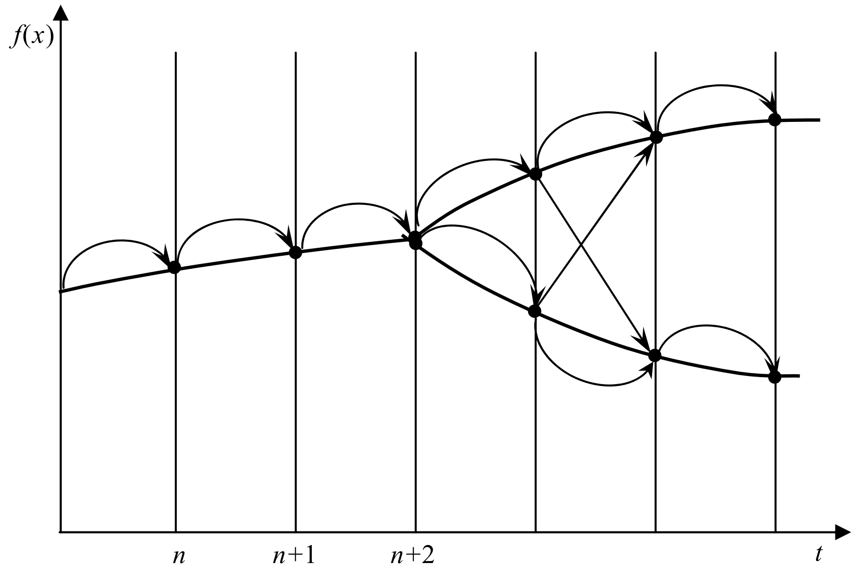

Fig. 6. General cases for an arbitrary transition function with possible dynamic transitions between two branches

A simpler case appears to be implemented for the function $\sqrt{\varphi}$, presumably with no jumps between branches, which requires further research. In this case, the idea of approximation with a finite set of values by branches can also be applied.

However, for arbitrary analytical functions case may be exist with a large number of branches (or even an infinite number of branches). In this case, all studies on the limit behavior of CA remain valid. However, limit behavior must be investigated for a multi-valued case, for which the theory of multi-valued dynamic systems should be involved [12]. Along the way, we will point out another new class of research problems. For ordinary traditional cellular automata (including the game of Life) we can set the following problem: according to the well-known rules of CA, to find CA with a complex-valued transition function such that traditional CA is an approximation of complex-valued CA.

In this case, it is possible to raise the problem of studying traditional CA. Probably, the function $\sqrt{\varphi}$ can only evolve over two branches independently of each other (without jumping from branch to branch). At the same time, in the case of an arbitrary function it is possible and the situation with jumps between branches is possible.

This creates a whole new set of problems associated with finding special function φ. The first class of problems is related to the construction related to a specific CA (including traditional ones). For some CA, it is possible to find the exact function by deforming known complex functions. Another possibility is to determine on the basis of known approaches from artificial intelligence: for example, identification methods or machine learning. But, it may turn out that precise definition of a function is very resource-intensive or unconstructive. Then it is necessary to simplify problem definition and achieve only the proximity of a given CA solution with a certain complex function $\widehat{\varphi}$ at a certain time range [0, T]. Closely related to this is the question of the stability of the behaviour of solutions to such approximation with a small change in function φ. It is also possible to use functions φ with an infinite number of branching points. It is also important to construct a function with given branching points so that these branching points set a fixed behavior of the solution or at least approximate the required behavior of a multi-valued solution on a certain range.

The next class of tasks related to the selection of functionality for the task is to build an algorithm for solving a particular task or with the theory of calculations. Note that possible links with quantum computing are viewed here. It is also possible to introduce a special category of complex-valued CA with branches.

The problems above are easy to generalize for other problems as well. Thus, nothing limits the transfer from the case of 1D space with only one branching point in one cell to the case of possible branching points in each cell of the cell space. Everything said above is carried over to the case of 2D, 3D, ND spaces. But it is also possible to generalize the very concept of cellular automata. Thus, it is possible to consider cell automata on Riemann surfaces. A particular problem is the study of the dependence of the dynamics of CA on the topology of the space in which CA is given.

By the way, the transition from CA with a continuum of possible values to approximations of finite sets correlates with the study of continuous dynamical systems with the methods of symbolic dynamics in ergodic theory.

CONCLUSIONS

Thus in proposed paper we had been introduced new class of cellular automata with presumable multivaluedness. New research problems are proposed for such cellular automata. The most important problem is the problem of approximation. Also further more complex research problem are described including the investigation of CA on the Riemann surfaces.

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INFORMATION ON THE ARTICLE

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МОДЕЛІ КЛІТИННИХ АВТОМАТІВ З КОМПЛЕКСНОЗНАЧНИМИ ФУНКЦІЯМИ ПЕРЕХОДУ / О.С. Макаренко

Анотація. Розглянуто новий клас математичних моделей для теорії обчислень — клітинні автомати (КА) з розгалуженими комплекснозначними перехідними функціями. Ключовим моментом є можлива багатозначність станів клітини з такими перехідними функціями. Розглянуто різні випадки з перехідними функціями з комплексними значеннями. Описано динаміку КА на одній гілці та на різних ізольованих гілках. Запропоновано розгляд випадку переходів станів між гілками, а також випадку неперервнозначних КА та їх кінцевозначних наближень. Поставлено проблему апроксимації багатозначних КА.

Ключові слова: клітинні автомати, комплекснозначні перехідні функції, поверхні Рімана, неперервнозначні КА, розгалуження, наближення багатозначності, обчислення.

МОДЕЛИ КЛЕТОЧНЫХ АВТОМАТОВ С КОМПЛЕКСНОЗНАЧНЫМИ ФУНКЦИЯМИ ПЕРЕХОДА / А.С. Макаренко

Аннотация. Рассмотрен новый класс математических моделей теории вычислений — клеточные автоматы (КА) с ветвящимися комплекснозначными переходными функциями. Ключевым моментом является возможная многозначность состояний клетки с такими переходными функциями. Рассмотрены различные случаи с комплексными переходными функциями. Описана динамика КА на одной ветке и на разных изолированных ветвях. Предложено рассмотрение случая переходов состояний между вервями, а также случая непрерывнозначных КА и их конечнозначных приближений. Поставлена проблема аппроксимации многозначных КА.

Ключевые слова: клеточные автоматы, комплекснозначные переходные функции, римановы поверхности, непрерывнозначные КА, ветвление, аппроксимация многозначности, вычисления.