UDC 519.711
DOI: 10.20535/SRIT.2308-8893.2023.2.07

GUARANTEED ROOT-MEAN-SQUARE ESTIMATES OF THE FORECAST OF MATRIX OBSERVATIONS UNDER CONDITIONS OF STATISTICAL UNCERTAINTY

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Abstract. We investigate the problem of linear estimation of unknown mathematical expectations based on observations of realizations of random matrix sequences. Constructive mathematical methods have been developed for finding linear guaranteed RMS estimates of unknown non-stationary parameters of average values based on observations of realizations of random matrix sequences. It is shown that such guaranteed estimates are obtained either as solutions to boundary value problems for systems of linear differential equations or as solutions to the corresponding Cauchy problems. We establish the form and look for errors for the guaranteed RMS quasi-minimax estimates of the special forecast vector and parameters of unknown average values. In the presence of small perturbations of known matrices in the model of matrix observations, quasi-minimax RMS estimates are found, and their guaranteed RMS errors are obtained in the first approximation of the small parameter method. Two test examples for calculating the guaranteed root mean square estimates and their errors are given.

Keywords: matrix observations, linear estimations, guaranteed RMS estimates, guaranteed RMS estimate errors, quasi-minimax guaranteed vector estimates, difference equation, small parameter method, matrix perturbation.

INTRODUCTION

This article examines estimates of unknown mathematical expectations based on observations of realizations of random matrix sequences. Scientific publications [1–14], in which estimates of distribution parameters were studied, are devoted to the problems of matrix sequence statistics. We formulate and solve new problems of estimating the mean values of random matrix sequences. Under the condition that the mean values belong to sets of a special form, we have developed constructive algorithms for guaranteed root-mean-square estimates of the mean values. It is shown that such estimates can be obtained either as solutions of boundary value problems for a system of linear differential equations, or as solutions of the corresponding Cauchy problems. In the case of the dependence of the average values on a small parameter, asymptotic distributions were obtained both for the guaranteed estimates and for the guaranteed root mean square errors of such estimates.

STATEMENT OF THE PROBLEM

We consider matrix observations of the form:

$$Y_k = \rho_k(x(k)) + \eta_k, \quad k = \overline{0, N-1}, \qquad (1)$$

where $\rho_k(x(k)) = \sum_{s=1}^{m} A_s(k)x_s(k), \ k = \overline{0, N-1}$,

$A_s(k) \in H_{n\times p}, \quad s = \overline{1, m}, \quad k = \overline{0, N-1}$ are known matrices;

$H_{n\times p}$ is the space of matrices $n \times p$ dimensions;

$x(k) = (x_1(k),...,x_m(k))^T, \ k = \overline{0, N}$ are unknown vectors, belonging to a limited set

$$G = \{x(k), k = \overline{0, N} : \sum_{k=0}^{N-1} |f(k)|^2 q_k^2 \le 1\},$$

$$f(k) = x(k+1) - x(k), \ k = \overline{0, N-1},$$

(to simplify the calculations, we assume that $x(0)$ is known vector and, without limiting the generality, we put $x(0) = 0$);

$q_k^2, \ k = \overline{0, N-1}$ are known positive real numbers;

T is a transposition symbol;

$\eta_k \in H_{n\times p}, \quad k = \overline{0, N-1}$ is a sequence of random matrices.

It is assumed that the average value of the random matrices η_k, $k = \overline{0, N-1}$ is equal to the null matrix, i.e. $E\eta_k = 0$ (E is a symbol of mathematical expectation), and correlation matrices $R_k \in H_{n\times n}, \ k = \overline{0, N-1}$ are known and determined by relations

$$E\langle \eta_k, Z_1\rangle\langle \eta_k, Z_2\rangle = \langle R_k Z_1, Z_2\rangle, \forall Z_i \in H_{n\times p}, \quad i = \overline{1,2}, \ k = \overline{0, N-1},$$

where $\langle \eta_k, Z_i\rangle = sp(\eta_k Z_i^T)$ is a scalar product of matrices.

Let's introduce linear operators that act from space R^l into space $H_{n\times p}$:

$$\overline{\rho}_k(U_k)(z) = \sum_{i=1}^{l} U_{ik} z_i, \ k = \overline{0, N-1},$$

$$U_{ik} \in H_{n\times p}, \quad U_k = (U_{1k} \vdots \ldots \vdots U_{lk}), \quad z = (z_1, \ldots, z_l)^T, \quad i = \overline{1, l}, k = \overline{0, N-1},$$

and operators conjugated to them $\overline{\rho}_k^*(U_k)(Y_k)$:

$$\overline{\rho}_k^*(U_k)(Y_k) = (\langle U_{1k}, Y_k\rangle, \ldots, \langle U_{lk}, Y_k\rangle)^T,$$

$$U_{ik} \in H_{n\times p}, \ Y_k \in H_{n\times p}, \ i = \overline{1, l}, \ k = \overline{0, N-1}.$$

It is necessary to evaluate the vector $Vx(N)$, where $V \in H_{l\times m}$.

Definition 1. A vector $\widehat{Vx}(N)$ of the form

$$\widehat{Vx}(N) = \sum_{k=0}^{N-1} \overline{\rho}_k^*(U_k)(Y_k) + c =$$

$$=\sum_{k=0}^{N-1}(\langle U_{1k},Y_k\rangle,\ldots,\langle U_{lk},Y_k\rangle)^T+c,\quad c\in R^l$$

is called a linear estimate of a vector $Vx(N)$.

Definition 2. The value

$$\sigma^2(U_0,\ldots,U_{N-1})=\max_G E\left[Vx(N)-\widehat{Vx}(N)\right]^2$$

is called *the guaranteed root mean square* (*RMS*) *error of the linear estimate* $\widehat{Vx}(N)$.

SOLVING THE PROBLEMS OF LINEAR ESTIMATION OF THE FORECAST OF MATRIX OBSERVATIONS.

I. Let's introduce vectors $z(k)\in R^m$, $k=\overline{0,N}$, which are solutions of the difference equation:

$$z(k)=z(k+1)-\rho_k^*\overline{\rho_k}(U_k)(a),\ k=\overline{N-1,0},\ z(N)=V^Ta,\ a\in R^l, \tag{2}$$

where ρ_k^*, $k=\overline{0,N-1}$ are operators conjugated to ρ_k.

Denote by $z^i(k)$, $i=\overline{1,l}$ the solutions of the difference equation (2) at $a=e^i$, where e^i, $i=\overline{1,l}$ are the base vectors of space R^l, and also enter the matrix Z:

$$Z=(z_{ij})_{i,j=\overline{1,l}};\quad z_{ij}=\sum_{k=0}^{N-1}(z^i(k+1),z^j(k+1))q_k^{-2};\ i,j=\overline{1,l}. \tag{3}$$

The vectors $z^i(k+1)$, $i=\overline{1,l}$ finds from the difference equations:

$$z^i(k)=z^i(k+1)+b^i(k),\quad z^i(N)=V_{(i)}, \tag{4}$$

where $b^i(k)=(\langle A_1(k),U_{ik}\rangle,\ldots,\langle A_m(k),U_{ik}\rangle)^T$,

$$V_{(i)}=(V_{i1},\ldots,V_{im})^T,\qquad i=\overline{1,l},\ k=\overline{0,N-1}.$$

There is a formula

$$z^i(k+1)=V_{(i)}+\sum_{j=1}^{N-(k+1)}b^i(N-j),\ i=\overline{1,l},\ k=\overline{0,N-1}. \tag{5}$$

Statement 1. Let $x(k)$, $k=\overline{0,N}\in G$, then the following equality holds:

$$\sigma^2(U_0,\ldots,U_{N-1})=\max_{|a|=1}((Za,a)^{1/2}+|(c,a)|)^2+\sum_{k=0}^{N-1}\sum_{i=1}^{l}\langle R_kU_{ik},U_{ik}\rangle.$$

Proof. Fair equality:

$$E\left|Vx(N)-\widehat{Vx}(N)\right|^2=\left|(Vx(N))-\sum_{k=0}^{N-1}(\overline{\rho}_k^*(U_k)(\rho_k(x(k)))-c\right|^2+$$

$$+E\left|\sum_{k=0}^{N-1}\overline{\rho}_k^*(U_k)\eta_k\right|^2=$$

$$=\max_{|a|=1}\left((V^Ta,x(N))-\left(a,\sum_{k=0}^{N-1}\overline{\rho}_k^*(U_k)(\rho_k(\rho_k(x(k))))-(a,c)\right)\right)^2+$$

$$+\sum_{k=0}^{N-1}\sum_{i=1}^{l}\langle R_kU_{ik},U_{ik}\rangle=$$

$$=\max_{|a|=1}\left((V^Ta,x(N))-\sum_{k=0}^{N-1}(x(k),\rho_k^*\overline{\rho}_k(U_k)(a))-(a,c)\right)^2+$$

$$+\sum_{k=0}^{N-1}\sum_{i=1}^{l}\langle R_kU_{ik},U_{ik}\rangle.$$

Since

$$(V^Ta,x(N))=(z(N),x(N))=\sum_{k=0}^{N-1}((z(k+1),x(k+1))-(z(k),x(k)))=$$

$$=\sum_{k=0}^{N-1}((z(k+1)-z(k)),x(k))+\sum_{k=0}^{N-1}(z(k+1),f(k)),$$

then, considering that $z(k+1)-z(k)=\rho_k^*\overline{\rho}_k(U_k)(a)$ we get that

$$\sigma^2(U_0,\ldots,U_{N-1})=\max_G E\left|Vx(N)-\widehat{Vx}(N)\right|^2=$$

$$=\max_{|a|=1}\max_G(\sum_{k=0}^{N-1}\big(z(k+1),f(k)\big)-(a,c))^2+\sum_{k=0}^{N-1}\sum_{i=1}^{l}\langle R_kU_{ik},U_{ik}\rangle=$$

$$=\max_{|a|=1}((\sum_{k=0}^{N-1}|z(k+1)|^2q_k^{-2})^{1/2}+|(a,c)|)^2+\sum_{k=0}^{N-1}\sum_{i=1}^{l}\langle R_kU_{ik},U_{ik}\rangle.$$

From the fact that equalities are fulfilled

$$(\sum_{k=0}^{N-1}|z(k+1)|^2q_k^{-2})=\sum_{i,j=1}^{l}(\sum_{k=0}^{N-1}(z^i(k+1),z^j(k+1))(a,e^i)(a,e^j)q_k^{-2})=(Za,a),$$

we conclude that the statement 1 is correct.

Corollary 1. There is an equality:

$$\min_{U,c}\max_G E\left|Vx(N)-\widehat{Vx}(N)\right|^2=\lambda_{\max}(\hat{Z})+\sum_{k=0}^{N-1}\sum_{i=1}^{l}\langle R_k\hat{U}_{ik},\hat{U}_{ik}\rangle,\ \hat{c}=0,$$

where $\hat{Z}=Z=(z_{ij})_{i,j=\overline{1,l}}$, $z_{ij}=z_{ij}(\hat{U})$, $\lambda_{\max}(\hat{Z})$ is the maximum eigenvalue of the matrix $\hat{Z}$, and $\hat{U}_{ik}\in H_{n\times p}$, $i=\overline{1,l}$, $k=\overline{0,N-1}$ are found from the condition:

$$(\hat{U}_{ik},i=\overline{1,l},\ k=\overline{0,N-1};\hat{c})\in \operatorname{Arg}\min_{c;U_{ik},i=\overline{1,l},k=\overline{0,N-1}}\sigma^2(U_0,\ldots,U_{N-1}).$$

Corollary 2. Let $l=1$. The estimation error $(V,x(N))$, where $V\in R^m$ is an arbitrary vector, is as follows:

$$\sigma^2(U_0,\ldots,U_{N-1})=\max_G E\left|Vx(N)-\widehat{Vx}(N)\right|^2=$$

$$=\sum_{k=0}^{N-1}\left|z(k+1)\right|^2 q_k^{-2}+\sum_{k=0}^{N-1}\langle R_k U_{1k},U_{1k}\rangle+c^2,$$

where $z(k)$, $k=\overline{0,N}$ is a solution of the difference equation $z(k)=$ $=z(k+1)-\rho_k^*(U_{1k})$, $k=\overline{N-1,0}$, $z(N)=V$, for this case $\hat{c}=0$.

Statement 2. Let's put the parameter $l=1$ in the statement 1, then:

1) guaranteed root mean square estimate for $\widehat{V_{(1)}x(N)}$ has the form:

$$\overline{\overline{V_{(1)}x(N)}}=\sum_{k=0}^{N-1} sp(\hat{U}_{1k}Y_k^T);$$

2) the guaranteed root mean square error of the linear estimate $\widehat{V_{(1)}x(N)}$ has the form:

$$\sigma^2(\hat{U}_{10},...,\hat{U}_{1,N-1})=(p(N),V_{(1)}),$$

where $\hat{U}_{1k}=R_k^+\rho_k(p(k))$, $k=\overline{0,N-1}$; R_k^+ is a pseudo-inverse operator;

$p(k)$, $k=\overline{0,N}$ are vectors that are determined from the system of equations:

$$\begin{cases} z(k)=z(k+1)-\rho_k^*\left(\hat{U}_{1k}\right), \ k=\overline{N-1,0}, \quad z(N)=V_{(1)}, \\ p(k+1)=p(k)+q_k^{-2}z(k+1), \ k=\overline{0,N-1}, \quad p(0)=0, \end{cases} \tag{6}$$

Proof. Let's define $\hat{U}_{1k}$ $k=\overline{0,N-1}$ from conditions:

$$\frac{d}{d\tau}\sigma^2(U_0+\tau\tilde{\vartheta}_0,...,U_{N-1}+\tau\tilde{\vartheta}_{N-1})_{\tau=0}=0, \quad \text{for } \forall\,\tilde{\vartheta}_k,\ k=\overline{0,N-1}.$$

There is an equality:

$$\frac{1}{2}\frac{d}{d\tau}\sigma^2(U_0+\tau\tilde{\vartheta}_0,...,U_{N-1}+\tau\tilde{\vartheta}_{N-1})_{\tau=0}=$$

$$=\sum_{k=0}^{N-1}(z(k+1),\tilde{z}(k+1))q_k^{-2}+\sum_{k=0}^{N-1}\langle R_k U_k,\tilde{\vartheta}_k\rangle,$$

where $\tilde{z}(k)=\tilde{z}(k+1)-\rho_k^*(\tilde{\vartheta}_k)$, $\tilde{z}(N)=0$, $k=\overline{0,N-1}$.

If input the vectors $p(k)\in R^m$, $k=\overline{0,N}$, which are solutions of difference equations

$$p(k+1)=p(k)+q_k^{-2}z(k+1), \ k=\overline{0,N-1}, \quad p(0)=0,$$

then we will get:

$$\sum_{k=0}^{N-1}(z(k+1),\tilde{z}(k+1))q_k^{-2}=\sum_{k=0}^{N-1}((\tilde{z}(k+1),(p(k+1)-p(k)))=$$

$$=\sum_{k=0}^{N-1}((\tilde{z}(k)-\tilde{z}(k+1)),p(k))=$$

$$=-\sum_{k=0}^{N-1}(\rho_k^*(\tilde{\vartheta}_k),p(k))=-\sum_{k=0}^{N-1}\langle\rho_k(p(k)),\tilde{\vartheta}_k\rangle.$$

As a result, we get equality:

$$\sum_{k=0}^{N-1}\langle(-\rho_k(p(k))+R_k U_{1k}),\tilde{\vartheta}_k\rangle=0,$$

from which the representation derives

$$\hat{U}_{1k} = R_k^+ \rho_k(p(k)), \ \ k = \overline{0, N-1},$$

which had to be shown.

Solution of the boundary value problem (6)

One of the options for solving the boundary value problem (6) is the possibility of reducing it to the solution of the Cauchy problem for vectors $p(k), \ k = \overline{0, N}$. This requires solving the Cauchy problem for the first equation of system (6). Then, after substituting the result into the second equation of the system, solve the Cauchy problem for the required vectors $p(k), \ k = \overline{0, N}$.

It is also possible to use the homogeneity of the considered problem for the required vectors $p(k), \ k = \overline{0, N}$. This requires the use of a base $e^i, \ i = \overline{1, m}$ of the vector space R^m. Expansions of vectors $z(k), \ \ p(k), \ k = \overline{0, N}$ in this basis have the form:

$$z(k) = \sum_{i=1}^m x_i z_i(k), \quad p(k) = \sum_{i=1}^m x_i p_i(k), \quad k = \overline{0, N}, \qquad (7)$$

where the vectors $p_i(k), \ z_i(k), \ k = \overline{0, N}, \ i = \overline{1, m}$ are defined as solutions of m boundary problems:

$$\begin{cases} z_i(k+1) = z_i(k) - F_k \, p_i(k), \ \ k = \overline{0, N-1}, \ \ z_i(0) = e^i; \\ p_i(k+1) = p_i(k) + q_k^{-2} z_i(k+1), \ k = \overline{0, N-1}, \ \ p_i(0) = 0, \ \ i = \overline{1, m}, \end{cases}$$

where

$$F_k = \rho_k^* R_k^+ \rho_k = \begin{pmatrix} sp(A_1^T R_k^+ A_1) & \cdots & sp(A_1^T R_k^+ A_m) \\ \vdots & \vdots & \vdots \\ sp(A_m^T R_k^+ A_1) & \cdots & sp(A_m^T R_k^+ A_m) \end{pmatrix}.$$

Unknown coefficients $x_i, \ i = \overline{1, m}$ in the expansions (7) are found as solutions of the system of linear algebraic equations that ensure the fulfillment of the boundary condition $z(N) = V$:

$$\sum_{i=1}^m z_i(N) x_i = V.$$

According to the distribution of vectors $p(k), \ k = \overline{0, N}$ (formula (7)), the expressions for the matrices $\hat{U}_{1k} = R_k^+ \rho_k(p(k)), \ \ k = \overline{0, N-1}$ of the required estimate $\widehat{Vx}(N)$ are obtained.

Another possibility of solving the boundary value problem (6) is to reduce it to a difference boundary value problem of the second order with respect to vectors $p(k), \ k = \overline{0, N}$ and to find a general solution to the obtained problem. Arbitrary constants of the general solution are determined from the boundary conditions of problem (6).

II. Let’s introduce vectors $\hat{p}(k)$, $\hat{x}(k), \quad k = \overline{0, N}$, that are the solutions of the system of difference equations:

$$\begin{cases} \hat{p}(k) = \hat{p}(k+1) + \rho_k^* R_k^+ (Y_k - \rho_k(\hat{x}(k))), \ \ \hat{p}(N) = 0; \\ \hat{x}(k+1) = \hat{x}(k) + q_k^{-2} \hat{p}(k+1), \qquad \hat{x}(0) = 0, \ \ k = \overline{0, N-1}. \end{cases} \qquad (8)$$

Remark 1. If we take into account that the equality holds

$$\rho_k^* R_k^+ (Y_k - \rho_k(\hat{x}(k))) = (sp(A_1^T R_k^+ Y_k),...,sp(A_m^T R_k^+ Y_k))^T - F_k \hat{x}(k),$$

then we can find the solution of linear differential equation system (8) according to the solution scheme of linear differential equation system (6).

Statement 3. The following equality holds

$$\overline{\overline{Vx}}(N) = (\hat{x}(N), V). \tag{9}$$

Proof. For a guaranteed estimate, the following relations are fulfilled:

$$\overline{\overline{Vx}}(N) = \sum_{k=0}^{N-1} \langle \hat{U}_k,\ Y_k \rangle = \sum_{k=0}^{N-1} \langle R_k^+ \rho_k(p(k)),\ Y_k \rangle = \sum_{k=0}^{N-1} (p(k), \rho_k^*(R_k^+ Y_k)). \tag{10}$$

Let's denote $\Delta_k = \hat{p}(k) - \hat{p}(k+1),\ k = \overline{0, N-1}$. Then

$$\rho_k^*(R_k^+ Y_k) = \Delta_k + \rho_k^*(R_k^+ \rho_k(\hat{x}(k))).$$

Hence

$$(p(k), \rho_k^*(R_k^+ Y_k)) = (p(k), \Delta_k) + (p(k), \rho_k^*(R_k^+ \rho_k(\hat{x}(k)))).$$

Now we sum up both parts of the last equality:

$$\sum_{k=0}^{N-1} (p(k), \rho_k^*(R_k^+ Y_k)) = \sum_{k=0}^{N-1} (p(k), \Delta_k) + \sum_{k=0}^{N-1} (p(k), \rho_k^*(R_k^+ \rho_k(\hat{x}(k)))) \tag{11}$$

and calculate the first term on the right-hand side:

$$\sum_{k=0}^{N-1} (p(k), \Delta_k) = \sum_{k=0}^{N-1} (p(k), \hat{p}(k) - \hat{p}(k+1)) =$$

$$= \sum_{k=0}^{N-1} (\hat{p}(k+1), p(k+1) - p(k)) = \sum_{k=0}^{N-1} (\hat{x}(k+1) - \hat{x}(k), z(k+1)) =$$

$$= \sum_{k=0}^{N-1} (\hat{x}(k), z(k) - z(k+1)) + (\hat{x}(N), V) =$$

$$= (\hat{x}(N), V) - \sum_{k=0}^{N-1} (\rho_k^*(R_k^+ \rho_k(p(k))), \hat{x}(k)) =$$

$$= (\hat{x}(N), V) - \sum_{k=0}^{N-1} (p(k), \rho_k^*(R_k^+ \rho_k(\hat{x}(k)))). \tag{12}$$

The required equality (9) follows from formulas (10)–(12).

Remark 2. The system of equations (8) can be obtained by solving the minimization problem of the function

$$J(f(0), \ldots, f(N-1)) =$$

$$= \sum_{k=0}^{N-1} \langle R_k^+ (Y_k - \rho_k(x(k))), Y_k - \rho_k(x(k)) \rangle + \sum_{k=0}^{N-1} q_k^2 f^2(k).$$

III. Below we consider the case when the set G is in the space of possible values $(x(0), \ldots, x(N))$, $x(k) = (x_1(k),...,x_m(k))^T$, $k = \overline{0, N}$ is unbounded and has the form:

$$G = \{(x(0), \ldots, x(N)) : \sum_{k=0}^{N-1} |x(k+1) - x(k)|^2 q_k^2 \le 1,\ \ x(0) \in R^m\},$$

where $q_k^2,\ k=\overline{0,N-1}$ are known positive real numbers.

It is necessary to determine the guaranteed root mean square error:

$$\sigma^2(U_0,\dots,U_{N-1})=\max_G E\left|Vx(N)-\widehat{Vx}(N)\right|^2,$$

where $\widehat{Vx}(N)=\sum_{k=0}^{N-1}(\langle U_{1k},Y_k\rangle,\dots,\langle U_{lk},Y_k\rangle)^T+c,\quad c\in R^l$.

Let's introduce vectors $z(k)\in R^l\ \ k=\overline{0,N-1}$, that are solutions of difference equations

$$z(k)=z(k+1)-\overline{\rho}_k^*\overline{\rho}_k(U_k)(a),\ \ z(N)=a,\quad k=\overline{0,N},$$

and the set $U=\{U_{ik},\ i=\overline{1,l},\ k=\overline{0,N-1}:z(0)=0\}$.

Statement 4.

1) If $U_{ik},i=\overline{1,l},\ k=\overline{0,N-1}\in U$, then

$$\sigma^2(U)=\max_G E\left|Vx(N)-\widehat{Vx}(N)\right|^2=$$

$$=\max_{|a|=1}((Za,a)^{1/2}+|(a,c)|)^2+\sum_{k=1}^{N-1}\sum_{i=1}^{l}\langle R_kU_{ik},U_{ik}\rangle. \qquad (13)$$

2) If $U_{ik},i=\overline{1,l},\ k=\overline{0,N-1}\notin U$, then

$$\sigma^2(U)=\max_G E\left|Vx(N)-\widehat{Vx}(N)\right|^2=\infty.$$

Proof.

1) If $U_{ik},\ i=\overline{1,l},\ k=\overline{0,N-1}\in U$, then we obtain the formula (13) similarly to the statement 1;

2) If $U_{ik}\notin U$, then there may exist $\overline{a}$ such that

$$\overline{z}(0)=z(0)_{a=\overline{a}}\neq 0 \text{ by } \forall U_{ik},\quad i=\overline{1,l},\ k=\overline{0,N-1}.$$

Therefore, given unbounded of the set G, we obtain the relation:

$$\sigma^2(U)\geq\max_G\ (\sum_{k=0}^{N-1}(\overline{z}(k+1),f(k))+(\overline{z}(0),x(0))-(\overline{a},c))^2+$$

$$+\sum_{k=0}^{N-1}\sum_{i=1}^{l}\langle R_kU_{ik},U_{ik}\rangle=\infty.$$

IV. We present a guaranteed linear RMS estimate of the scalar product $(a,x(N))$ according to matrix observations of the form

$$Y_k=\rho_k(x(k))+\eta_k,\quad k=\overline{0,N-1},$$

through the solutions of the Cauchy problem for linear differential equations.

Denote by $V_k,\ k=0,1,2,...$ the sequence of linear operators of the form:

$$V_k=(R_k+\rho_kP_{k-1}\rho_k^*)\rho_kP_{k-1},$$

where matrices P_k are solutions of difference equations:

$$P_k = P_{k-1} + V_k^*(R_k + \rho_k P_{k-1}\rho_k^*)V_k + Q_k,$$

$$Q_k = q_k^{-2} I_m,\;\; P_{-1} = 0,\;\; k = 0,1,2,...$$

Statement 5. The equality holds

$$\widehat{(a, x(N))} = \sum_{k=0}^{N-1} \langle \hat{U}_k, Y_k \rangle, \quad \hat{U}_k = V_k\, z(k+1),$$

where $z(k),\; k = \overline{1,N}$ are the solutions of difference equations:

$$z(k) = z(k+1) - V_k z(k+1) \equiv (I_m - V_k) z(k+1),\; k = \overline{N,1},\;\; z(N) = a.$$

Moreover

$$\max_G E((a, x(N)) - \widehat{(a, x(N))})^2 = (P_{N-1} a, a).$$

Proof. Let's solve the problem of optimal system control:

$$z(k) = z(k+1) - \rho_k^*(U_k), \quad z(N) = a,\; k = \overline{N-1,0}$$

with the criterion

$$\bar{J}(U_0,...,U_{N-1}) = \sum_{k=0}^{N-1} (Q_k z(k+1), z(k+1)) + \sum_{k=0}^{N-1} \langle R_k U_k, U_k \rangle$$

by the method of dynamic programming.

Let's introduce the Bellman function

$$B_k(x) = \min_{U_0,...,U_k} [\sum_{i=0}^{k} (Q_i z(i+1), z(i+1)) + \sum_{i=0}^{k} \langle R_i U_i, U_i \rangle],\;\; z(k+1) = x,$$

for which the Bellman equation holds

$$B_k(x) = (Q_k x, x) + \min_{u \in H_{n\times p}} [B_k(x)(x - \rho_k^*(u)) + \langle R_k u, u \rangle],\;\; B_{-1}(x) = 0.$$

$$k = \overline{-1, N-1}$$

(by definition we consider that $\sum_{k=0}^{-1} = 0$).

We find the function $B_k(x)$ in the form $B_k(x) = (P_k x, x)$. Let's choose the matrices $P_k,\; k = \overline{-1, N-1}$ so that the Bellman equation holds true. After appropriate transformations (similarly as it is done, for example, in [15]) we obtain the expressions for $\hat{U}_k,\; V_k,\; P_k,\;\; k = \overline{0, N-1}$.

It is obvious that $B_{N-1}(x) = (P_{N-1} a, a) = \max_G (E((a, x(N)) - \widehat{(a, x(N))})^2$.

Statement 6. For a guaranteed linear RMS estimate of the scalar product the following representation takes place

$$\widehat{(a, x(N))} = (a, \hat{x}(N)),$$

where the vector $\hat{x}(N)$ is a solution of difference equations

$$\hat{x}(k+1) = \hat{x}(k) + V_k^*(Y_k - \rho_k(\hat{x}(k)),\; k = \overline{0, N-1},\;\; \hat{x}(0) = 0. \quad (14)$$

Proof. Since equalities are fulfilled:

$$\hat{U}_k = V_k z(k+1), \quad k = \overline{0, N-1},$$

then $\sum_{k=0}^{N-1} \langle \hat{U}_k, Y_k \rangle = \sum_{k=0}^{N-1} (z(k+1), V_k^* Y_k)$.

Expressions take place:

$$V_k^* Y_k = \hat{x}(k+1) - \hat{x}(k) + V_k^* \rho_k(\hat{x}(k));$$

$$\sum_{k=0}^{N-1} (z(k+1), V_k^* Y_k) = \sum_{k=0}^{N-1} (z(k+1), \hat{x}(k+1) - \hat{x}(k)) +$$

$$+ \sum_{k=0}^{N-1} (z(k+1), V_k^* \rho_k(\hat{x}(k)));$$

$$\sum_{k=0}^{N-1} (z(k+1), \hat{x}(k+1) - \hat{x}(k)) = -\sum_{k=0}^{N-1} (z(k+1) - z(k), \hat{x}(k)) + (a, \hat{x}(N)) =$$

$$= (a, \hat{x}(N)) - \sum_{k=0}^{N-1} (\rho_k^* (V_k z(k+1)), \hat{x}(k)) =$$

$$= (a, \hat{x}(N)) - \sum_{k=0}^{N-1} (z(k+1), V_k^* \rho_k(\hat{x}(k))).$$

From here we get the necessary equality.

Remark 3. The vector $\hat{x}(N)$ is found as a solution of the linear difference equation (14). It is possible to obtain the vector $\hat{x}(N)$ even if the vectors $f(k) = \hat{x}(k+1) - \hat{x}(k)$, $k = \overline{0, N-1}$ are random and uncorrelated ($Ef(k) = 0$, $Ef(k) f^T(k) = q_k^2$, $k = \overline{0, N-1}$. The given estimators are such that minimize the root mean square error in the category of linear estimators.

V. **Definition 3.** The vector

$$\widetilde{Vx}(N) = \sum_{k=0}^{N-1} (\langle \hat{U}_{1k}, Y_k \rangle, \ldots, \langle \hat{U}_{lk}, Y_k \rangle)^T,$$

which components are calculated according to formulas

$$\hat{U}_{ik} = R_k^+ \rho_k(p_{(i)}(k)), \quad i = \overline{1, l}, \quad k = \overline{0, N-1},$$

and $p_{(i)}(k)$ are vectors that are determined from the systems of difference equations

$$\begin{cases} z_{(i)}(k) = z_{(i)}(k+1) - \rho_k^*(\hat{U}_{ik}), & z_{(i)}(N) = V_{(i)}, \\ p_{(i)}(k+1) = p_{(i)}(k) - q_k^{-2} z_{(i)}(k+1), & p_{(i)}(0) = 0, \end{cases}$$

$$i = \overline{1, l}, \quad k = \overline{0, N-1}.$$

is called the quasi-minimax guaranteed estimation of the vector

$$\widehat{Vx}(N) = \sum_{k=0}^{N-1} (\langle U_{1k}, Y_k \rangle, \ldots, \langle U_{lk}, Y_k \rangle)^T.$$

Statement 7. For the guaranteed root mean square error of quasi-minimax estimates there is equality:

$$\sigma^2(\hat{U}_0, \ldots, \hat{U}_{N-1}) = \lambda_{\max}(\hat{Z}) + \sum_{k=0}^{N-1} \sum_{i=1}^{l} \langle R_k \hat{U}_{ik}, \hat{U}_{ik} \rangle,$$

where $\hat{Z} = (\hat{z}_{ij})_{i,j=\overline{1,m}}$, $\hat{z}_{ij} = \sum_{k=0}^{N-1} (\hat{z}_i(k+1), \hat{z}_i(k+1)) q_k^{-2}$, $i, j = \overline{1, m}$, and elements $\hat{z}_i(k+1)$, $i = \overline{1, m}$ are found as solutions of difference equations:

$$\hat{z}_i(k) = \hat{z}_i(k+1) + \hat{b}_i(k),\ k = \overline{N-1,1},\ \ \hat{z}_i(N) = V_{(i)},$$

$$\hat{b}_i(k) = (\langle A_1(k), \hat{U}_{ik}\rangle, \ldots, \langle A_m(k), \hat{U}_{ik}\rangle)^T,$$

$$V_{(i)} = (V_{i1}, \ldots, V_{im})^T, \qquad i = \overline{1,l},\ k = \overline{1, N-1}.$$

Finding elements $\hat{z}_{ij},\ i, j = \overline{1,m}$ of the matrix $\hat{Z}$ is carried out according to the algorithm for calculating elements of matrix Z (formulas (3)–(5)).

Quasi-minimax RMS estimates for small matrix perturbations.

Assume that the known matrices of model (1) have the form:

$$A_s(k) = A_s^{(0)}(k) + \varepsilon A_s^{(1)}(k) \in H_{n\times p},\quad s = \overline{1,m},\ k = \overline{0, N-1},$$

where $\varepsilon \in R^1$ is small parameter; and the operators are as follows:

$$\rho_k^{(\varepsilon)}(x(k)) = \rho_k^{(0)}(x(k)) + \varepsilon \rho_k^{(1)}(x(k)),$$

where $\rho_k^{(0)}(x(k)) = \sum_{s=1}^m A_s^{(0)}(k) x_s(k),\ \rho_k^{(1)}(x(k)) = \sum_{s=1}^m A_s^{(1)}(k) x_s(k),\ k = \overline{0, N-1}.$

We determine the effect of small perturbations of the matrices on the estimates, as well as on their errors, using the results presented in statement 3.

Statement 8. Quasi-minimax guaranteed estimation of the vector $Vx(N)$ within the framework of the first approximation of the small parameter method has the form:

$$\widetilde{Vx}(N) = \sum_{k=0}^{N-1} (\langle \hat{U}_{1k}^{(\varepsilon)}, Y_k\rangle, \ldots, \langle \hat{U}_{lk}^{(\varepsilon)}, Y_k\rangle)^T,$$

where $\hat{U}_{ik}^{(\varepsilon)} = \hat{U}_{ik}^{(0)} + \varepsilon \hat{U}_{ik}^{(1)} + o(\varepsilon) I_{n\times p}$,

$I_{n\times p} \in H_{n\times p}$ is the matrix, all elements of which are equal to one,

$$\hat{U}_{ik}^{(0)} = R_k^+ \rho_k^{(0)}(p_{(i)}^{(0)}(k)),$$

$$\hat{U}_{ik}^{(1)} = R_k^+ \rho_k^{(0)}(p_{(i)}^{(1)}(k)) + R_k^+ \rho_k^{(1)}(p_{(i)}^{(0)}(k)),\ i = \overline{1,l},\ k = \overline{0, N-1},$$

and vectors $p_{(i)}^{(0)}(k),\ i = \overline{1,l},\ k = \overline{0,N}$ are defined as solutions of boundary value problems:

$$z_{(i)}^{(0)}(k) = z_{(i)}^{(0)}(k+1) - \rho_k^{(0)^*}(R_k^+ \rho_k^{(0)}(p_{(i)}^{(0)}(k))), \qquad z_i^{(0)}(N) = V_{(i)},$$

$$p_{(i)}^{(0)}(k+1) = p_{(i)}^{(0)}(k) + q_k^{-2} z_{(i)}^{(0)}(k+1),\ \ p_i^{(0)}(0) = 0,\quad i = \overline{1,l},\ \ k = \overline{0, N-1},$$

$$z_{(i)}^{(1)}(k) = z_{(i)}^{(1)}(k+1) - \rho_k^{(0)^*}(R_k^+ \rho_k^{(0)}(p_{(i)}^{(1)}(k))) -$$

$$-\rho_k^{(1)^*}(R_k^+ \rho_k^{(0)}(p_{(i)}^{(0)}(k))) - \rho_k^{(0)^*}(R_k^+ \rho_k^{(1)}(p_{(i)}^{(0)}(k))),\ \ z_{(i)}^{(1)}(N) = 0,$$

$$p_{(i)}^{(1)}(k+1) = p_{(i)}^{(1)}(k) + q_k^{-2} z_{(i)}^{(1)}(k+1),\ \ p_{(i)}^{(1)}(0) = 0,\quad i = \overline{1,l},\ \ k = \overline{0, N-1}.$$

Statement 9. There is the equality for the guaranteed root mean square error of quasi-minimax estimates within the framework of the first approximation of the small parameter method:

$$\sigma^2(\hat{U}_0^{(\varepsilon)},\ldots,\hat{U}_{N-1}^{(\varepsilon)}) = \lambda_{\max}(\hat{Z}^{(\varepsilon)}) + \sum_{k=0}^{N-1}\sum_{i=1}^{l}\langle R_k \hat{U}_{ik}^{(\varepsilon)}, \hat{U}_{ik}^{(\varepsilon)}\rangle.$$

Here $\hat{U}_{ik}^{(\varepsilon)} = \hat{U}_{ik}^{(0)} + \varepsilon\hat{U}_{ik}^{(1)} + o(\varepsilon)I_{n\times p}$, $\quad i=\overline{1,l},\ k=\overline{0,N-1}$, and the expression for the matrix $\hat{Z}^{(\varepsilon)}$ has the form

$$\hat{Z}^{(\varepsilon)} = \hat{Z}^{(0)} + \varepsilon\hat{Z}^{(1)} + o(\varepsilon)I_{m\times m},$$

where $\hat{Z}^{(0)} = (\hat{z}^{(0)}{}_{i,j})_{i,j=\overline{1,m}}$, $\hat{z}_{i,j}^{(0)} = \sum_{k=0}^{N-1}(\hat{z}_i^{(0)}(k+1), \hat{z}_j^{(0)}(k+1))q_k^{-2}$;

$$\hat{Z}^{(1)} = (\hat{z}^{(1)}{}_{i,j})_{i,j=\overline{1,m}},\quad \hat{z}_{i,j}^{(1)} = 2\sum_{k=0}^{N-1}(\hat{z}_i^{(0)}(k+1), \hat{z}_j^{(1)}(k+1))q_k^{-2}.$$

The vectors $\hat{z}_i^{(0)}(k+1)$, $i=\overline{1,m},\ k=\overline{0,N-1}$ are found as solutions of zero-approximation difference equations:

$$\hat{z}_i^{(0)}(k) = \hat{z}_i^{(0)}(k+1) + \hat{b}_i^{(0)}(k),\quad z_i^{(0)}(N) = V_{(i)}; \tag{15}$$

$$\hat{b}_i^{(0)}(k) = \left(\langle A_1^{(0)}(k), \widehat{U_{ik}^{(0)}}\rangle,\ldots,\langle A_m^{(0)}(k), \widehat{U_{ik}^{(0)}}\rangle\right)^T;$$

$$V_{(i)} = (V_{i1},\ldots,V_{im})^T,\quad i=\overline{1,l},\ k=\overline{0,N-1},$$

and vectors $\hat{z}_j^{(1)}(k+1)$, $j=\overline{1,m},\ k=\overline{0,N-1}$ are found as solutions of first approximation difference equations:

$$\hat{z}_j^{(1)}(k) = \hat{z}_j^{(1)}(k+1) + \hat{b}_j^{(1)}(k),\quad z_j^{(1)}(N) = 0; \tag{16}$$

$$\hat{b}_j^{(1)}(k) = (\langle A_1^{(1)}(k), \hat{U}_{jk}^{(0)}\rangle,\ldots,\langle A_m^{(1)}(k), \hat{U}_{jk}^{(0)}\rangle)^T +$$
$$+ (\langle A_1^{(0)}(k), \hat{U}_{jk}^{(1)}\rangle,\ldots,\langle A_m^{(0)}(k), \hat{U}_{jk}^{(1)}\rangle)^T,\quad j=\overline{1,l},\ k=\overline{0,N-1}.$$

Finding the solutions of differential equations (15), (16) is carried out according to the algorithm for calculating elements of matrix $Z = (z_{ij})_{i,j=\overline{1,l}}$ (formulas (3)–(5)).

Example 1. Let the matrix observations have the form:

$$Y_k = \rho_k^{(\varepsilon)}(x(k)) + \eta_k,\quad k=\overline{0,N}; \tag{17}$$

$$\rho_k^{(\varepsilon)}(x(k)) = \rho_k^{(0)}(x(k)) + \varepsilon\rho_k^{(1)}(x(k)), \tag{18}$$

where $\rho_k^{(0)}(x(k)) = A^{(0)}x(k)$, $\quad \rho_k^{(1)}(x(k)) = A^{(1)}(k)x(k)$,

$$A^{(0)} = I_2,\quad A^{(1)}(k) = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix},\ k=\overline{0,N},\ \varepsilon\in R^1 \text{ is a small parameter;}$$

$x(k)\in R^1$, $k=\overline{0,N-1}$ and belong to a bounded set G:

$$G = \{x(k),\ k = \overline{0,N} : \sum_{k=0}^{N-1} |x(k+1) - x(k)|^2 \le q_1^{-2} \};$$

$\eta_k \in H_{2\times 2}$, $k = \overline{0,N}$ is a sequence of random matrices.

It is assumed that the average value of the random matrices η_k, $k = \overline{0,N-1}$ are equal to the zero matrix, and is the correlation matrices $R_k = q_0^2 I_2$, $k = \overline{0,N-1}$.

The guaranteed RMS estimate $\hat{x}(N)$ has the form:

$$\hat{x}(N) = \sum_{k=0}^{N-1} \langle \hat{U}_k^{(\varepsilon)}, Y_k \rangle, \tag{19}$$

where $\hat{U}_k^{(\varepsilon)} = q_0^{-2} \rho_k^{(\varepsilon)}(p^{(\varepsilon)}(k))$, $k = \overline{0,N-1}$;

$p^{(\varepsilon)}(k)$, $k = \overline{0,N}$ are values that are determined from the system of difference equations:

$$\begin{cases} z^{(\varepsilon)}(k) = z^{(\varepsilon)}(k+1) - q_0^2 \langle A^{(\varepsilon)}(k), A^{(\varepsilon)}(k) \rangle p^{(\varepsilon)}(k),\ k = \overline{N-1,1},\ z^{(\varepsilon)}(N) = 1; \\ p^{(\varepsilon)}(k+1) = p^{(\varepsilon)}(k) + q_1^2 z^{(\varepsilon)}(k+1),\ k = \overline{0,N-1},\ p^{(\varepsilon)}(0) = 0. \end{cases} \tag{20}$$

Applying the small parameter method for solving problems (19), (20), we introduce the following notation:

$$\hat{U}_k^{(\varepsilon)} = \widehat{U_k}(0) + \varepsilon \hat{U}_k(1) + o(\varepsilon) I_{2\times 2}, \quad k = \overline{0,N-1},$$

$$\widehat{U_k}(0) = q_0^{-2} \rho_k^{(0)}(p^{(0)}(k)),\ \widehat{U_k}(1) = q_0^{-2}(\rho_k^{(0)}(p^{(1)}(k)) + \rho_k^{(1)}(p^{(0)}(k))),$$

$$p^{(\varepsilon)}(k) = p^{(0)}(k) + \varepsilon p^{(1)}(k) + o(\varepsilon), \quad z^{(\varepsilon)}(k) = z^{(0)}(k) + \varepsilon z^{(1)}(k) + o(\varepsilon),$$

where $p^{(0)}(k)$, $k = \overline{0,N}$ are the values of the zero approximation of the small parameter method, which are defined as solutions of the boundary value problem:

$$\begin{cases} z^{(0)}(k) = z^{(0)}(k+1) - \beta p^{(0)}(k),\ z^{(0)}(N) = 1, \\ p^{(0)}(k+1) = z^{(0)}(k) + q_1^2 z^{(0)}(k+1), \quad p^{(0)}(0) = 0, \end{cases} \tag{21}$$
$$\beta = 2q_0^{-2}, \quad k = \overline{0,N-1},$$

and $p^{(1)}(k)$, $k = \overline{0,N}$ are the values of the first approximation of the small parameter method, which are defined as solutions of the boundary value problem:

$$\begin{cases} z^{(1)}(k) = z^{(1)}(k+1) - p^{(1)}(k) - \beta k\, p^{(0)}(k), \quad z^{(1)}(N) = 0, \\ p^{(1)}(k+1) = p^{(1)}(k) + q_1^2 z^{(1)}(k+1), \quad p^{(1)}(0) = 0,\ k = \overline{0,N-1}. \end{cases} \tag{22}$$

Solution of the boundary value problem (21) of zero approximation

The boundary value problem (21) reduces to a boundary value problem for values $p^{(0)}(k)$, $k = \overline{0,N}$:

$$p^{(0)}(k+2) - (2+q) p^{(0)}(k+1) + p^{(0)}(k) = 0, \qquad p^{(0)}(0) = 0,$$
$$p^{(0)}(N) - p^{(0)}(N-1) = q_1^2, \tag{23}$$

$$k=\overline{0,N-2},\ \ q=2q_1^2q_0^{-2}.$$

The general solution of the corresponding homogeneous system has the form:

$$p_0^{(0)}(k)=C_1\lambda^k+C_2\lambda^{-k},\quad \lambda=\left(2+q+\sqrt{q^2+4q}\right)/2.$$

Taking into account the boundary conditions, the solution of the boundary value problem (23) is as follows:

$$p_0^{(0)}(k)=C_1(\lambda^k-\lambda^{-k}),\quad C_1=q_1^2\lambda^N/((\lambda-1)(\lambda^{2N-1}+1)).$$

Therefore, in the zero approximation of the small parameter method, expressions are obtained:

$$\widehat{U_k}(0)=q_0^{-2}p^{(0)}(k),\quad k=\overline{0,N-1}. \tag{24}$$

The guaranteed RMS estimate in the zero approximation of the small parameter method has the form:

$$\hat{x}^{(0)}(N)=q_0^{-2}C_1\ \sum_{k=0}^{N-1}\langle A^{(0)},Y_k\rangle(\lambda^k-\lambda^{-k}), \tag{25}$$

and the error of this estimate is as follows:

$$\max_G\{E[x(N)-\hat{x}(N)]^2\}^{1/2}=\{p^{(0)}(N)\}^{1/2}, \tag{26}$$

where $p^{(0)}(N)=\dfrac{q_1^2(\lambda-1/\lambda^{2N-1})}{(\lambda-1)(\lambda+1/\lambda^{2N-1})}$.

The representation of the estimation error in the zero approximation by formula (26) allows one to notice a decrease in its value with an increase in the quantity of observations, as well as to establish a limit value $p^{(0)}(N)$:

$$\lim_{N\to\infty}p^{(0)}(N)=q_1^2\,\frac{(-1+\sqrt{1+2q_1^{-2}q_0^2})}{2}.$$

Solving the boundary value problem (22) of the first approximation

The boundary value problem (22) is reduced to a boundary value problem for values $p^{(1)}(k),\ k=\overline{0,N}$:

$$p^{(1)}(k+2)-p^{(1)}(k+1)+p^{(1)}(k)=\beta k(\lambda^k-\lambda^{-k}), \tag{27}$$

$$p^{(1)}(0)=0,\ \ p^{(1)}(N)-p^{(1)}(N-1)=0,\ \ k=\overline{0,N-2}.$$

The partial solution of the inhomogeneous equation (27) is represented by formulas with undefined coefficients:

$$p_{\text{part}}^{(1)}(k)=(B_1k^2+B_2k+B_3)\lambda^k+(D_1k^2+D_2k+D_3)\lambda^{-k},\quad k=\overline{0,N},$$

which have the form:

$$B_1=\beta/(\lambda^2-1),\ \ B_2=-B_1(3\lambda^2-1)/(\lambda^2-1),\ \ B_3=0,$$

$$D_1=-B_1,\quad D_2=B_2,\ \ D_3=0.$$

The general solution of the inhomogeneous equation (27) has the form:

$$p^{(1)}(k) = F_1\lambda^k + F_2\lambda^{-k} + (B_1k^2 + B_2k)(\lambda^k - \lambda^{-k}), \quad k = \overline{0,N},$$

where arbitrary constants are determined from boundary conditions (27).

The solution of the boundary value problem (27) is as follows:

$$p^{(1)}(k) = [F_1 + (B_1k^2 + B_2k)](\lambda^k - \lambda^{-k}), \; k = \overline{0,N},$$

where $F_1 = \dfrac{1}{(\lambda-1)}\left\{[B_1(N-1)+B_2](N-1) - (B_1N + B_2\lambda)N\dfrac{\lambda-(1/\lambda^{2N-1})}{1-(1/\lambda^{2N-1})}\right\}$.

Thus, expressions are obtained for the corrections of the first approximation of the small parameter method:

$$\hat{U}_k(1) = q_0^{-2}(A^{(0)}p^{(1)}(k) + A^{(1)}(k)p^{(0)}(k)), \quad k = \overline{0,N-1},$$

$$\hat{x}^{(1)}(N) = \sum_{k=0}^{N-1}\langle \hat{U}_k(1), Y_k\rangle,$$

$$p^{(1)}(N) = [F_1 + (B_1N^2 + B_2N)](\lambda^N - \lambda^{-N}). \tag{28}$$

The guaranteed root mean square error in the first approximation of the small parameter method is represented by the formula:

$$\sigma^2(\varepsilon) = \sigma^2(0) + \varepsilon\sigma^2(1) + o(\varepsilon),$$

where

$$\sigma^2(0) = \frac{q_1^2}{(\lambda-1)}\frac{(\lambda-(1/\lambda^{2N-1}))}{(1+(1/\lambda^{2N-1}))},$$

$$\sigma^2(1) = p^{(1)}(N) = (F_1 + (B_1N^2 + B_2N))(\lambda^N - \lambda^{-N}).$$

Remark 4. It is worth noting that when using formula (28), it is necessary to take into account the specific values of the model parameters of the observation problem $q_0, q_1, q, \lambda,$ as well as the number of observations N, namely: order of magnitude $\varepsilon\sigma^2(1)$ a smaller than order of magnitude $\sigma^2(0)$.

The extended possibilities of applying the small parameter method can be seen in the following example for other small perturbations of the known matrices in the model of the observation problem.

Example 2. Let the matrix observations have the form represented by formulas (17), (18), but with other matrices of small perturbation:

$$A^{(1)}(k) = a^{-k}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; k = \overline{0,N-1}. \tag{29}$$

Let's find the decomposition for the small parameter of the guaranteed value estimate $x(N)$, as well as its errors.

As in example 1, the guaranteed RMS estimate $\hat{x}(N)$ has the form:

$$\hat{x}(N) = \sum_{k=0}^{N-1}\langle \hat{U}_k^{(\varepsilon)}, Y_k\rangle, \tag{30}$$

where $\hat{U}_k^{(\varepsilon)} = q_0^{-2}\rho_k^{(\varepsilon)}(p^{(\varepsilon)}(k)), \; k = \overline{0,N-1}$; $p^{(\varepsilon)}(k), \; k = \overline{0,N}$ are the values that are determined from the system of difference equations:

$$\begin{cases} z^{(\varepsilon)}(k) = z^{(\varepsilon)}(k+1) - q_0^2 \langle A^{(\varepsilon)}(k), A^{(\varepsilon)}(k)\rangle p^{(\varepsilon)}(k), \ k = \overline{N-1,1}, \ z^{(\varepsilon)}(N) = 1, \\ p^{(\varepsilon)}(k+1) = p^{(\varepsilon)}(k) + q_1^2 z^{(\varepsilon)}(k+1), \ k = \overline{0, N-1}, \ p^{(\varepsilon)}(0) = 0. \end{cases} \quad (31)$$

Applying the small parameter method to solve problems (30), (31) we obtain a guaranteed root mean square estimate in the zero approximation and its error in the formulas form (24)–(26).

First approximation corrections of the small parameter method for the guaranteed RMS estimate require the definition of matrices $\hat{U}_k(1) = = q_0^{-2}(\rho_k^{(0)}(p^{(1)}(k)) + \rho_k^{(1)}(p^{(0)}(k)))$, where $p^{(0)}(k),\ k = \overline{0, N}$ are the values of the zero approximation of the small parameter method, and $p^{(1)}(k),\ k = \overline{0, N}$ are the values of the first approximation of the small parameter method, which are defined as solutions of boundary value problems

$$p^{(1)}(k+2) - p^{(1)}(k+1) + p^{(1)}(k) = \beta a^{-k}\ (\lambda^k - \lambda^{-k}), \quad (32)$$

$$p^{(1)}(0) = 0,\ p^{(1)}(N) - p^{(1)}(N-1) = 0,\ k = \overline{0, N-2}.$$

The partial solution of the inhomogeneous equation (27) is represented by formulas with undefined coefficients:

$$p_{\text{part}}^{(1)}(k) = B_1 \left(\frac{\lambda}{a}\right)^k + B_2 (a\lambda)^{-k}, \quad k = \overline{0, N},$$

where these coefficients are calculated by formulas:

$$B_1 = \beta a^2 / (\lambda^2 - \lambda a + a^2), \quad B_2 = -\beta a^2 \lambda^2 / (a^2 \lambda^2 - \lambda a + 1).$$

The general solution of the inhomogeneous equation (27) has the form:

$$p^{(1)}(k) = F_1 \lambda^k + F_2 \lambda^{-k} + B_1 \left(\frac{\lambda}{a}\right)^k + B_2 (a\lambda)^{-k}, \quad k = \overline{0, N},$$

and arbitrary constants $F_1,\ F_2$ are determined from the boundary conditions (32):

$$F_1 = -(F_2 + B_1 + B_2),$$

$$F_2 = \frac{1}{(1 + 1/\lambda^{-2N+1})} \left[B_1 \left(a^{-N} \frac{(\lambda - a)}{(\lambda - 1)} - 1 \right) + B_2 \left(a^{-N} \lambda^{-2N+1} \frac{(\lambda a - 1)}{(\lambda - 1)} + 1 \right) \right].$$

Thus, expressions are obtained for the corrections of the first approximation of the small parameter method:

$$\hat{U}_k(1) = q_0^{-2}(A^{(0)} p^{(1)}(k) + A^{(1)}(k) p^{(0)}(k)), \quad k = \overline{0, N-1},$$

$$\hat{x}^{(1)}(N) = \sum_{k=0}^{N-1} \langle \hat{U}_k(1), Y_k \rangle,$$

$$p^{(1)}(N) = F_1 \lambda^N + F_2 \lambda^{-N} + B_1 \left(\frac{\lambda}{a}\right)^N + B_2 (a\lambda)^{-N}.$$

The guaranteed root mean square error in the first approximation of the small parameter method is represented by the formula:

$$\sigma^2(\varepsilon) = \sigma^2(0) + \varepsilon \sigma^2(1) + o(\varepsilon),$$

where $\sigma^2(0) = \frac{q_1^2}{(\lambda-1)} \frac{(\lambda - (1/\lambda^{2N-1}))}{(1+(1/\lambda^{2N-1}))}$,

$$\sigma^2(1) = p^{(1)}(N) = F_1\lambda^N + F_2\lambda^{-N} + B_1\left(\frac{\lambda}{a}\right)^N + B_2(\lambda a)^{-N}.$$

Obviously, that at certain values of the parameter a in model (29), the desired accuracy of the small parameter method can be achieved with a larger number of observations.

CONCLUSIONS

The article develops constructive mathematical methods for finding linear guaranteed root mean square estimates of unknown non-stationary parameters of average values based on observations of realizations of a sequence of random matrices. It is shown that, under certain conditions, such estimates are expressed in terms of solutions of the boundary value problem for the system of difference equations. Formulas are presented that allow obtaining recurrent estimates of unknown parameters. In the case of the dependence of the average values on a small parameter, the corresponding asymptotic formulas are given. Asymptotic distributions of linear parameter estimates and their root mean square errors are given for partial cases.

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Received 20.01.2023

INFORMATION ON THE ARTICLE

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ГАРАНТОВАНІ СЕРЕДНЬОКВАДРАТИЧНІ ОЦІНКИ ПРОГНОЗУ МАТРИЧНИХ СПОСТЕРЕЖЕНЬ В УМОВАХ СТАТИСТИЧНОЇ НЕВИЗНАЧЕНОСТІ / О.Г. Наконечний, Г.І. Кудін, П.М. Зінько, Т.П. Зінько

Анотація. Досліджено задачу лінійного оцінювання невідомих математичних сподівань за спостереженнями реалізацій випадкових матричних послідовностей. Розроблено конструктивні математичні методи для знаходження лінійних гарантованих середньоквадратичних оцінок невідомих нестаціонарних параметрів середніх значень за спостереженнями реалізацій послідовності випадкових матриць. Показано, що такі гарантовані оцінки одержуються або як розв'язки крайових задач для систем лінійних різницевих рівнянь, або як розв'язки відповідних задач Коші. Установлено вигляд похибок для гарантованих середньоквадратичних квазімінімаксних оцінок спеціального вектора прогнозу та параметрів невідомих середніх значень. За наявності малих збурень відомих матриць у моделі матричних спостережень знайдено квазімінімаксні середньоквадратичні оцінки і в першому наближенні методу малого параметра отримано їх гарантовані середньоквадратичні похибки. Наведено два тестові приклади обчислення гарантованих середньоквадратичних оцінок та їх похибок.

Ключові слова: матричні спостереження, лінійне оцінювання, гарантована середньоквадратична оцінка, похибка гарантованої середньоквадратичної оцінки, квазімінімаксна гарантована оцінка вектора, різницеве рівняння, метод малого параметра, збурення матриць.