

GENERALIZATION OF THE THERMODYNAMIC APPROACH TO MULTI-DIMENSIONAL QUASISTATIC PROCESSES

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Abstract. A method of mathematical modeling of multidimensional quasi-static processes, a generalization of quasi-static processes of equilibrium thermodynamics, is proposed and substantiated. The authors obtain a generalization of the first and the second law of thermodynamics in the form of Carathéodory to multidimensional quasi-static processes. The idea of generalization is to construct an orthogonal system of functionals similar to the work and heat functionals of classical thermodynamics along families of phase trajectories corresponding to different types of influences on a multidimensional quasi-static system. The representation of quasi-static processes by systems of ordinary differential equations containing control variables is substantiated. The obtained results make it possible to use a wide arsenal of methods of the theory of control of dynamical systems to solve problems of control of quasi-static processes.

Keywords: quasi-static processes, equilibrium thermodynamics, mathematical modeling, work, energy, controllability, entropy.

INTRODUCTION

The concepts of “quasi-static system” and “quasi-static process” owe their appearance to the fundamental branch of physics – thermodynamics. Thermodynamics originated and was further developed on the basis of the phenomenological approach, i.e. considering only the observed properties of thermodynamic systems (TS) without a detailed analysis of the mechanisms of their manifestation. As a result of this approach to the formation of thermodynamics, its main provisions: terminology, axiomatics and logical constructions in the form of physical laws have not changed significantly over the past few centuries. In the same time real practical results confirming theoretical backgrounds with a sufficiently high degree of accuracy have made thermodynamics, along with mechanics, a model of applied science. The quasi-static nature of processes in TS is one of the main postulates of thermodynamics, along with such concepts as reversibility and equilibrium. From the standpoint of thermodynamics, a quasi-static process is an infinitely slow process consisting of a sequence of equilibrium states infinitely close to each other. An essential limitation of the thermodynamics of equilibrium processes is the exclusion from consideration of the time coordinate and the representation of processes in the form of segments of trajectories in the phase space.

The mathematical support of thermodynamics is based on two postulates: the first and second laws, on the basis of which mathematical models of the main thermodynamic processes are introduced.

Despite the apparent limitation of the object of study in the form of quasi-static TS, a number of concepts and approaches of thermodynamics can be adapted to a relatively wide class of technological systems similar in external properties to TS. In this regard, the problem of constructing a mathematical theory of quasi-static processes arises, covering both the TS and other systems with the quasi-static property. In this article, an attempt to propose the mathematical foundations of the theory of quasi-static processes with a focus on the control problem is made.

REVIEW AND ANALYSIS OF INFORMATION SOURCES

The classical substantiation of thermodynamics as a system of knowledge is almost completely described in the works of Carnot, Clausius, Boltzmann, Planck [1–3] and other “classics” of thermodynamics [4, 5]. This branch of thermodynamics has survived without fundamental changes to the present day.

One of the principal ways to involve mathematical methods in thermodynamics and generalize the latter to other physical and technical processes can be considered the approach outlined in [6]. A feature of Gukhman’s ideas is the substantiation of the uniformity of the components of the differential equation of the first law of thermodynamics in the form of products of certain potentials and a change in the corresponding coordinate of the state of the TS. At the same time, the thermal effect was included as a particular type of the set of effects on the TS, and the concepts of energy and amount of effect received a generalized interpretation suitable for any similar processes. The versatility of the ideas presented in [6] is clearly demonstrated in studies in the field of ecosystems [7].

Summarizing the results of thermodynamics related to the first law or the law of conservation of energy, we can conclude that the mathematical methods that allow adapting classical equilibrium thermodynamics to a wide range of similar processes in nature, technology and society are limited. The fundamental disadvantage of the considered approach is the absence of any time dependences linking changes in the state of the TS with the intensity (power) of actions.

In this regard, the authors of this work previously obtained some results of the “dynamization” of quasi-static processes. This made it possible to obtain mathematical models in the form of ordinary differential equations relating the change in time of the state vector of a quasi-static system to changes in the power of control actions. The results obtained in [8–10] require further generalization.

The greatest difficulty in the formation of thermodynamics was caused by its second law, and in particular the concept of entropy. For instance, in [4] there are 18 formulations of the 2nd law, each of which ultimately has the same meaning of the impossibility of obtaining mechanical work using only one source of heat. Unfortunately, the abundance of formulations and interpretations of the 2nd law of thermodynamics has led to some conservatism in the mathematization and formalization of thermodynamic methods and their adaptation to quasi-static processes of a different nature. The works of Schiller, Carathéodory, Afanasieva-Ehrenfest, Belokon were a qualitative shift in the direction of axiomatization in

thermodynamics, where a generalized theory of quasi-static processes was formulated in the language of mathematical formulations. Among the studies in this direction, the article by Carathéodory [11] should be especially singled out, where he elegantly formulated the 2nd law of thermodynamics in the language of Pfaffian forms. This article gave impetus to the wide application of the axiomatic approach in the thermodynamics of multidimensional processes [12–15].

MATHEMATICAL MODEL OF A CONTROLLED QUASI-STATIC PROCESS

We will consider multidimensional quasi-static processes in terms similar to the terms of classical thermodynamics: state, action, phase trajectory, work, energy, etc. The fundamental difference from thermodynamics will be the inclusion of the concept of heat in the category of works, as proposed in [6]. Thus, in our further constructions, heat or other effects will not take any priority position among the many possible effects on the system under consideration.

Let the state of the system under consideration Σ is given by the state vector $x \in R^n$. State change over time $x(t)$ represents a trajectory Σ in $(n+1)$ -dimensional space. Any change of the vector x in time is associated with the presence of one or more actions from a given set W . In the absence of actions on a quasi-static system, the latter remains in the state in which all actions were interrupted. That is any condition $x \in R^n$ of a quasi-static system is an equilibrium position, in contrast to dynamical systems, for which the set of equilibrium positions is a subset of the space R^n .

Associate with each $m \leq n$ of the types of actions W_k a vector field $\varphi_k(x)$, $k = \overline{1, m}$. We will assume that the system of vector fields $\varphi_1(x), \varphi_2(x), \dots, \varphi_m(x)$ at every point $x \in R^n$ is linearly independent. For k -th action the trajectory Σ in phase space can be represented as a solution to the system of differential equations

$$\frac{dx}{dt} = \varphi_k(x)u_k(t), \quad (1)$$

where $u_k(t)$ is a scalar function of the parameter t , which has the meaning of the action intensity W_k . Absolute value $u_k(t)$ determines the speed of movement of the representative point along the phase trajectory, and its sign determines the direction of movement.

Let now several actions be simultaneously applied to the system. They correspond to the vector field $\varphi(x)$. Let's decompose the vector $\varphi(x)$ at some point x along linearly independent vectors $\varphi_1(x), \varphi_2(x), \dots, \varphi_m(x)$. We get

$$\varphi(x) = \sum_{k=1}^m \varphi_k(x)u_k,$$

where u_k are decomposition coefficients.

Then the movement of the system along an arbitrary phase trajectory corresponding to the complex action will correspond to the system of differential equations

$$\frac{dx}{dt} = \sum_{k=1}^m \varphi_k(x)u_k(t), \quad (2)$$

and the coefficients u_k can be interpreted as the intensity of the corresponding actions.

If time t is chosen as a parameter, and values $u_k(t)$ are considered as controls, then the mathematical model of the quasi-static process in the form (2) is a controlled system linear in controls. Since the vectors are linearly independent, then the equilibrium of system (2) is $\varphi_1(x), \varphi_2(x), \dots, \varphi_m(x)$ achieved only in the case $u_1 = u_2 = \dots = u_m = 0$. That is, the effects cannot compensate each other.

The proposed mathematical models of quasi-static processes with single (1) and complex action (2) give a qualitative model of behavior Σ . The point is that the vector fields $\varphi_k(x)$ ($k = \overline{1, m}$) specify only the directions of the corresponding fields and can be normalized arbitrarily. In this case, the form of phase trajectories will remain unchanged. At the same time, the trajectory Σ in time will depend both on the normalization of the vector $\varphi_k(x)$, as well as on the size $u_k(t)$, forming the speed of movement Σ along the phase trajectory defined by the vectors $\varphi_k(x)$ and some starting point x_0 of the specific process.

ENERGY MEASURE OF THE ACTION

The previously introduced concept of the action intensity, which has the meaning of controlling a quasi-static process, requires a quantitative measurement. To do this, as an analogy, we use the classical thermodynamic approach for thermal deformation systems, i.e., TS subject only to mechanical and thermal effects, the phase trajectories of which have the form $pV^\gamma = \text{const}$ and $p = \text{const}$ respectively, where p, V are the pressure and volume of the gas in relative units and γ is adiabatic exponent ($\gamma \approx 1.3 \div 1.5$), depending on the chemical composition of the substance.

It is known from thermodynamics [5] that the first law of thermodynamics is valid for a thermal deformation system, which is formulated as follows:

$$dU = \Delta L + \Delta Q, \quad (3)$$

where dU is an increment of internal energy, ΔL is a deformation work, ΔQ is a supplied heat. In (3) internal energy $U(x)$ is a function of state and quantities ΔL and ΔQ are functions of trajectories in a state space.

For ideal gases [5] the internal energy U has the form

$$U = \frac{1}{\gamma - 1} pV, \quad (4)$$

and the mechanical work and heat are calculated as curvilinear integrals along the phase trajectory l :

$$\Delta L = - \int_{(l)} p dV; \quad (5)$$

$$\Delta Q = \int_{(l)} \left(\frac{1}{\gamma - 1} V dp + \frac{\gamma}{\gamma - 1} p dV \right). \quad (6)$$

Thus, ratio (5) and (6) can be interpreted as a quantitative measure of the action on the TS in the form of the mechanical work (5) and heat (6). For fixed initial x_0 and final x_f process points $U(x_f) - U(x_0) = \text{const}$ for any trajectories connecting x_0 and x_f , the sum ΔL and ΔQ in accordance with (3) will be constant for various ratios ΔL and ΔQ .

Let us generalize the thermodynamic approach (4) as applied to finding similar quantitative measures of actions for an arbitrary n -dimensional quasi-static process with $m = n$ actions. Let us introduce R^n the scalar continuous function in static space – the energy:

$$U(x) = U(x_1, x_2, \dots, x_n). \quad (7)$$

The energy increment (7) in some point $x \in R^n$ will look like:

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \dots + \frac{\partial U}{\partial x_n} dx_n. \quad (8)$$

Denoting $\frac{\partial U}{\partial x} \triangleq g(x)$ — the gradient $U(x)$, write down (8) as

$$dU = (g, dx), \quad (9)$$

and along the trajectory of the solution (1), or along the phase trajectory of the k -th action, the (8) takes the form

$$dU = (g, \varphi_k) dt. \quad (10)$$

Ratio (10) will also be valid for any phase trajectory generated by an arbitrary vector field $\varphi(x)$

$$dU = (g, \varphi) dt. \quad (11)$$

Let us denote the set of possible phase trajectory l_k corresponding to k -th type of action as L_k , and the set $\bigcup L_k \triangleq L$. The set of all feasible trajectories corresponding to an arbitrary composition of actions is denoted by L^* . Then there is the system of inclusions

$$L_k \subset L \subset L^*, \quad \forall k = \overline{1, m}.$$

We will call the energy measure $J(l)$ of any phase trajectory l the value

$$J(l) = \int_{(l)} (g, dx). \quad (12)$$

Due to (9) the energy measure (12) depends only on the initial x_0 and final x_f points of the curve l :

$$J(l) = U(x_f) - U(x_0) = J(x_0, x_f). \quad (13)$$

The energy measure (13) satisfies the following axioms:

$$1) J(x, x) = 0;$$

$$2) J(x, y) = -J(y, x); \quad (14)$$

$$3) J(x, y) = J(x, z) + J(z, y).$$

Axioms 2 and 3 in (14) differ from the symmetry and triangle axioms known from the theory of metric spaces:

$$J(x, y) = J(y, x);$$

$$J(x, y) \leq J(x, z) + J(z, y).$$

By analogy with first law of thermodynamics, we will look for the quantities of each type of actions in the form of functionals $A_k(l)$, calculated along phase trajectories similar to the mechanical work and heat for the simplest thermodynamic system (5), (6):

$$A_k(l) = \int_{(l)} (q_k(x), dx), \quad k = \overline{1, n}. \quad (15)$$

Functionals $A_k(l)$ we will call the works of action W_k . Vectors $q_k(x)$ we will be chosen so that following conditions are satisfied:

$$A_k(l) = \begin{cases} J(l), & \text{if } l \in L_k, \\ 0, & \text{if } l \in L_i, \end{cases} \quad i, k = \overline{1, n}, \quad i \neq k. \quad (16)$$

Requirements (16) will be met under the following conditions to choose from $q_k(x)$:

$$(q_k, \varphi_i) = \delta_{ik}(g, \varphi_i), \quad i, k = \overline{1, n}, \quad (17)$$

where δ_{ik} is the Kronecker symbol.

The conditions (17) are essentially conditions for the quasi-biorthogonality of vector systems $(\varphi_1, \varphi_2 \dots \varphi_n)$ and $(q_1, q_2 \dots q_n)$.

Rewrite the last equation in matrix form:

$$Q\Phi = \text{diag}(\Phi^T g), \quad (18)$$

where the rows of matrix Q are desired vectors $(q_1, q_2 \dots q_n)$ and the columns of the matrix Φ are vector fields $(\varphi_1, \varphi_2 \dots \varphi_n)$ of actions on the system.

Since vectors $\varphi_k(x)$ are assumed to be linearly independent for $\forall x \in R^n$, matrix Φ is non-degenerate, therefore the solution (18) can be written as

$$Q = \text{diag}(\Phi^T g)\Phi^{-1}. \quad (19)$$

Let us now show that for the system of works $A_1, A_2 \dots A_n$ with vector fields, defined in accordance with (19), $\forall l \in L^*$ the following relation is true:

$$J(l) = \sum_{k=1}^n A_k(l). \quad (20)$$

To prove (20), we sum relations (17) over k . As a result, we get:

$$\sum_{k=1}^n (q_k, \varphi_i) = \sum_{k=1}^n \delta_{ik}(g, \varphi_i), \quad i = \overline{1, n};$$

$$\left(\sum_{k=1}^n q_k, \varphi_i \right) = (g, \varphi_i), \quad i = \overline{1, n}.$$

It follows directly from the latter:

$$\sum_{k=1}^n q_k = g = \frac{\partial U}{\partial x}. \quad (21)$$

Let us sum over k the values $A_k(l)$, determined in accordance with (15). As a result, we get

$$\sum_{k=1}^n A_k(l) = \int_l \left(\sum_{k=1}^n q_k, dx \right). \quad (22)$$

Taking into account relations (12) and (21), and from (22) we get (20), i.e. the sum of the work calculated along an arbitrary trajectory is equal to the increment in the energy of the system at the ends of this trajectory.

As an example consider thermodynamic system with actions in the form of mass transfer, deformation, and heat transfer. The vector fields of such actions are well known [5] and are represented by the matrix $\Phi(p, V, m)$, looking like

$$\Phi = \begin{pmatrix} \gamma \frac{p}{m} & -\gamma \frac{p}{V} & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (23)$$

where p, V, m are pressure, volume and mass of the system. The columns of the matrix (22) are vector fields of mass transfer, deformation and thermal effects, respectively.

Let us introduce the energy of system in the form of a known relation for ideal gases (3), then

$$\frac{\partial U}{\partial x} = \left(\frac{1}{\gamma-1} V, \frac{1}{\gamma-1} p, 0 \right) = (g_1, g_2, g_3). \quad (24)$$

Substituting the (23) and (24) in the formula (19), we get a matrix

$$Q = \begin{pmatrix} 0 & 0 & \frac{\gamma}{\gamma-1} \frac{pV}{m} \\ 0 & -p & 0 \\ \frac{V}{\gamma-1} & \frac{\gamma p}{\gamma-1} & -\frac{\gamma}{\gamma-1} \frac{pV}{m} \end{pmatrix},$$

the rows of which are the components of the vectors of the quantities of the corresponding actions

$$A_1 = \int_{(l)} \frac{\gamma}{\gamma-1} \frac{pV}{m} dm; \quad A_2 = \int_{(l)} -pdV;$$

$$A_3 = \int_{(l)} \left(\frac{V}{\gamma-1} dp + \frac{\gamma}{\gamma-1} pdV - \frac{\gamma}{\gamma-1} \frac{pV}{m} dm \right),$$

where A_1 is a work of mass transfer, A_2 is a mechanical work, A_3 is a heat.

It is easy to verify that sum of the rows of the matrix Q coincides with the internal energy gradient, and

$$A_1(l) + A_2(l) + A_3(l) = J(l).$$

The considered approach to constructing a system of functionals satisfying the additivity condition (20) can also be extended to the case when the number of actions m is less than the dimension of the system n . In this case, the matrix Φ in the system of equations (18) is rectangular, and, consequently, constructions (19)–(24) lose their meaning. A way out of this situation can be obtained as a result of redefining the system of actions with the missing $n - m$ dummy actions. These dummy actions are chosen so that the square matrix $\Phi = (\Phi_1; \Phi_2)$, where $\Phi_1 - (n \times m)$ – the matrix given actions, and $\Phi_2 - (n \times (n - m))$ – the matrix of dummy actions will be nondegenerate. Substituting the matrix Φ obtained this way in relation (19) we get the matrix of works, which will depend on an arbitrary matrix Φ_2 .

INVERSE PROBLEM

The performed analysis showed that if the laws of change of the phase coordinates of the thermodynamic system depending on the type of influences are known, then for an arbitrarily given state function $U(x)$, a system of functionals can be chosen that satisfies the structure of the energy conservation law. The question arises, is it possible to solve the inverse problem: for a given functional corresponding to the work of some action, construct the corresponding function $U(x)$, which has the meaning of the energy of the thermodynamic system. This approach is quite pragmatic, since it allows to “join” various physical processes that have the same types of external influences. For example, in mechanics and thermodynamics, such an action is deformation, and the corresponding work in both cases is calculated by similar functionals:

$$A = - \int_{(l)} f ds \text{ is mechanics;}$$

$$L = - \int_{(l)} p dV \text{ is thermodynamics.}$$

So, let's formulate the problem as follows. Let some reference action be given, which corresponds to the phase trajectories of the system of differential equations

$$\dot{x} = \varphi(x), \quad x \in R^n,$$

where $\varphi(x)$ is a known vector function.

Let the known functional correspond to the standard action, with the help of which the work of the standard action is calculated:

$$L = \int_{(l)} q_1(x) dx_1 + q_2(x) dx_2 + \dots + q_n(x) dx_n = \int_{(l)} (q, dx).$$

Integrating the last differential equation, we obtain the n -th first integral of the original system of differential equations of characteristics (25) in the form

$$U + \lambda_n(x_n, C) = C_n. \quad (29)$$

Substituting into (29) the expressions for the components of the vector of constants C in the form (27), we finally obtain

$$U + \Psi_n(x) = C_n.$$

As is known from the theory of partial differential equations, the general solution $U(x)$ satisfies the equation

$$\Phi(U + \Psi_n(x), \Psi_1(x), \Psi_2(x), \dots, \Psi_{n-1}(x)) = 0,$$

where Φ is an arbitrary function of its arguments.

Solving the last equation with respect to the first argument, we finally obtain

$$U = U_0(x) + \Phi[\Psi_1(x), \Psi_2(x), \dots, \Psi_{n-1}(x)], \quad (30)$$

where $U_0(x) = -\Psi_n(x)$, and Φ is an arbitrary function of its arguments.

Let's analyze the result. As can be seen from (30), the function $U(x)$ that satisfies the differential equation for the operation of a given action consists of two terms. The first term $U_0(x)$ is determined both by the type of the phase trajectory of the system subject to the reference action and by the type of the standard functional (vector $q(x)$). The second term depends only on the phase trajectories and is an arbitrary function of the first integrals of the system of differential equations that model the behavior of the thermodynamic system under the reference action.

Thus, all $\Psi_i(x)$ $i = \overline{1, n-1}$ that are arguments of an arbitrary function Φ are constants, and therefore $\Phi[\Psi_1, \Psi_2, \dots, \Psi_{n-1}]$ also an arbitrary constant. Thus,

$$U(x) = U_0(x) + C,$$

i.e. the energy of a quasi-static system is determined up to some constant C .

Consider an example of finding the internal energy $U(p, V)$ for the simplest thermodynamic system subjected to a standard deformation effect, the work of which is defined as

$$L = - \int_{V_1}^{V_2} p dV.$$

The system of differential equations modeling the deformation effect, as follows from [5], has the form

$$\frac{dp}{dt} = -\gamma \frac{p}{V}, \quad \frac{dV}{dt} = 1,$$

and differential equation (24) for $U(p, V)$ can be written in the form

$$-\gamma \frac{p}{V} \frac{\partial U}{\partial p} + \frac{\partial U}{\partial V} = -p.$$

The corresponding system of ordinary differential equations of characteristics can be written as

$$\frac{dp}{-\gamma \frac{p}{V}} = \frac{dV}{1} = \frac{dU}{-p}. \quad (31)$$

The first integral $\Psi_1(p, V)$, which is the equation of the adiabatic phase curve, is obtained by integrating the differential equation

$$\frac{dp}{-\gamma \frac{p}{V}} = \frac{dV}{1}$$

and has a well-known form

$$pV^\gamma = C_1. \quad (32)$$

Expressing p through C_1 and V , and substituting the result into (31), we obtain a differential equation for determining the energy $U(p, V)$. Thus

$$p = C_1 V^{-\gamma}, \quad dU = -C_1 V^{-\gamma} dV, \quad U = -C_1 \int V^{-\gamma} dV, \quad \text{and} \quad U = -C_1 \frac{V^{-\gamma+1}}{1-\gamma} + C.$$

Substituting expression (32) for C_1 into the last relation, we finally obtain the expression, well known from thermodynamics for the internal energy of an ideal gas,

$$U = \frac{1}{\gamma-1} pV + C.$$

It should be noted that this result was obtained only on the basis of the known experimental adiabatic curves without involving the results of the well-known experiment of Gay-Lussac and Joule, which showed the invariance of the gas temperature during its expansion into a vacuum.

Let us now return to the original mathematical model of the thermodynamic system (2) and find the physical meaning of the quantities u_k – an action intensities of W_k . To do this, we proceed as follows: let the work A_j be given by an integral of the form:

$$A_j = \int_{(l)} (q_j, dx), \quad (33)$$

where q_j is previously defined vector of j -th work. Multiply the left and right parts of (2) scalarly by the vector q_j , as a result we get

$$(q_j, \dot{x}) = \sum_{k=1}^n (q_j, \Phi_k) u_k. \quad (34)$$

Taking into account the conditions of orthogonality of works (12), from relations (33)–(34) we get

$$\frac{dA_j}{dt} = (q_j, \Phi_j) u_j.$$

From the last expression, and also from (17) it follows

$$u_j = \frac{1}{(q_j, \varphi_j)} \dot{A}_j = \frac{1}{(g, \varphi_j)} \dot{A}_j, \quad j = \overline{1, n}. \quad (35)$$

Let us substitute (35) into original differential equation (2). As result we get

$$\frac{dx}{dt} = \sum_{k=1}^n \frac{\varphi_k(x)}{(g, \varphi_k)} \dot{A}_k. \quad (36)$$

From (36) it follows that if the column vectors $\varphi_k(x)$ of matrix $\Phi(x)$ divide accordingly into dot products (g, φ_k) , then each of the controls u_k will match the intensity of the corresponding work \dot{A}_k , i. e., using electromechanical terminology, u_k with this normalization corresponds to the action power W_k . In addition, the original system of equations (2) can be immediately normalized in accordance with (36), which will in no way affect the form of phase trajectories corresponding to certain types of action and the corresponding work vector $q_k(x)$.

Thus, without loss of generality, we can assume that the right side of the system of equations (2) is normalized in accordance with (34), and the control vector u_j corresponds to the power vector \dot{A}_j . Therefore, any of the works can be calculated both in the form of a curvilinear integral (33) and in the form of an integral over the time

$$A_j = \int_{t_0}^{t_1} u_j(t) dt.$$

QUASI-STATIC CONTROL SYSTEM OF GENERAL VIEW

Let us consider an empirical approach to the construction of mathematical models of quasi-static systems, given by families of phase trajectories L_1, L_2, \dots, L_n , corresponding to the impacts W_1, W_2, \dots, W_n . Let L_k it look like

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= C_1; \\ f_2(x_1, x_2, \dots, x_n) &= C_2; \\ &\dots\dots\dots \\ f_{n-1}(x_1, x_2, \dots, x_n) &= C_{n-1}, \end{aligned} \quad (37)$$

of $(n-1)$ -parametric family of lines in n -dimensional phase space. The set of constants C_1, C_2, \dots, C_{n-1} set a specific phase trajectory $l_k \in L_k$, and a change in one of the variables (for example, x_n) determines all the coordinates of the points of the selected phase trajectory as a result of solving system (37). Let's associate the change x_n with some parameter $t \in R^1$, which can be taken as time. Then all x_k will also be functions of time due to equations (37). To find the vector field corresponding to trajectories from L_k , we differentiate (37) with respect to t . As a result, we get

ing to Carathéodory, in the vicinity of an arbitrary state of the TS, there are adiabatically inaccessible states. In other words, in the absence of thermal action, the phase trajectories of the TS belong to some hypersurfaces, which are a one-parameter family

$$f(x) = c .$$

To carry out to the transition of the representative point from a hypersurface $f(x) = c_1$ on a hypersurface $f(x) = c_2$ it is necessary to involve thermal action. Carathéodory proved that in this case the differential form included in the integral (15) for the elementary amount of heat dQ is holonomic, i.e. must have an integrating factor $\psi(x)$:

$$dQ = (q(x), dx) = \psi(x)(\bar{q}(x), dx) = \psi(x)dS(x) .$$

Function $S(x)$, which remains constant in the absence of thermal interaction, corresponds to the entropy of the TS. In [17], general criteria for the holonomy of differential Pfaffian forms are formulated.

Thus, the second law of thermodynamics in the formulation of Carathéodory from the point of view of control theory is a criterion for the controllability of a n -dimensional dynamical system of the form (1) under a $n-1$ control action. Since this paper presents an abstract approach to equilibrium thermodynamics, the authors consider it fundamental to extend the Carathéodory method to an arbitrary number of control actions. In other words, below we will analyze the controllability of dynamic systems of the form (1) with the $m \leq (n-1)$ numbers of actions.

To solve the formulated problem, we study the structure of the set of phase trajectories of solution of differential equations corresponding to m action

$$\frac{dx}{dt} = \varphi_1(x), \dots, \frac{dx}{dt} = \varphi_m(x) . \quad (40)$$

Regarding the set of phase trajectories of m systems (40), we assume that it belongs to some invariant one-parameter manifold

$$S(x) = c , \quad (41)$$

i.e. all points $x(t)$ of phase trajectories of the set of systems (32), satisfy the condition

$$S(x(t)) = c , \quad \forall t \in [0, \infty) ,$$

where the constant c is determined from the initial condition

$$c = S(x(t_0)) .$$

In this case, the set of states, that do not satisfy condition (41) will be inaccessible for a given set of actions, and from the point of view of control theory, such a system will be uncontrollable.

To find the surface $S(x)$ we compose the system of differential equations in partial derivatives, which it must satisfy

$$\left(\frac{\partial S}{\partial x}, \varphi_1(x) \right) = 0, \dots, \left(\frac{\partial S}{\partial x}, \varphi_m(x) \right) = 0 . \quad (42)$$

The system (42) the conditions of orthogonality for the gradient vector of the function $S(x)$ and for all vector fields that determined the TS trajectories for a given structures of actions. Thus the problem of inaccessibility of a TS of an arbitrary set of actions is reduced to the analysis of the solvability of the system of equations (42), which is the system of linear differential equations in partial derivatives with respect to the function $S(x)$.

The theory of linear partial differential equations is quite well developed and contains a solution algorithm, including an analysis of the conditions for the existence of a solution [18]. This condition can be represented as

$$\text{rank}\{\varphi_1, \varphi_2, \dots, \varphi_m, [\varphi_1, \varphi_2], [v] \dots [\varphi_{m-1}, \varphi_m], [\varphi_1, [\varphi_2, \varphi_3]] \dots\} < n, \quad (43)$$

where $[\varphi, \psi]$ is the commutator of the vector fields φ and ψ .

$$[\varphi, \psi] = \frac{\partial \psi}{\partial x} \varphi - \frac{\partial \varphi}{\partial x} \psi.$$

Thus, this fulfilment of the condition (43) indicates that there is an invariant surface $S(x) = c$, to which all possible trajectories of the TS motion belong, i.e. (43) is a criterion for the uncontrollability of the generalized thermodynamic system.

THE DISCUSSION OF THE RESULTS

The thermodynamic approach to mathematical modeling of multidimensional quasi-static processes is generalized. The concepts of energy and work corresponding to each of the feasible actions on the controlled system are introduced. The conditions for the orthogonality of the vectors of the coefficients of the differentials of work made it possible to formulate the law of conservation of energy, similar to the first law of thermodynamics, in the most general form without involving any physical laws.

The axiomatic approach to the formation of the 2nd law of thermodynamics has been further developed as applied to multidimensional quasi-static systems. The concept of controllability of a quasi-static system is introduced.

Conditions for the existence of an invariant set of states of a quasi-static system are obtained for the number of actions less than the dimension of the system. It is shown that the existence of an invariant set is equivalent to the criterion of uncontrollability of a quasi-static system.

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УЗАГАЛЬНЕННЯ ТЕРМОДИНАМІЧНОГО ПІДХОДУ ДО БАГАТОВИМІРНИХ КВАЗІСТАТИЧНИХ ПРОЦЕСІВ / О.С. Куценко, С.В. Коваленко, С.М. Коваленко

Анотація. Запропоновано і обґрунтовано метод математичного моделювання багатовимірних квазістатичних процесів, які є узагальненням квазістатичних процесів рівноважної термодинаміки. Отримано узагальнення першого, а також другого закону термодинаміки у формі Каратеодорі на багатовимірні квазістатичні процеси. Ідея узагальнення — побудова ортогональної системи функціоналів, аналогічних функціоналам роботи і теплоти класичної термодинаміки уздовж сімей фазових траєкторій, що відповідають різним видам впливів на багатовимірну квазістатичну систему. Обґрунтовано подання квазістатичних процесів системами звичайних диференціальних рівнянь, що містять керувальні змінні. Отримані результати дозволяють залучити широкий арсенал методів теорії керування динамічними системами до розв'язання задач керування квазістатичними процесами.

Ключові слова: квазістатичні процеси, рівноважна термодинаміка, математичне моделювання, робота, енергія, керованість, ентропія.