UDC 004.942:537.525
DOI: 10.20535/SRIT.2308-8893.2023.3.10

BASIC ALGORITHM FOR APPROXIMATION OF THE BOUNDARY TRAJECTORY OF SHORT-FOCUS ELECTRON BEAM USING THE ROOT-POLYNOMIAL FUNCTIONS OF THE FOURTH AND FIFTH ORDER

I. MELNYK, A. POCHYNOK

Abstract. The new iterative method of approximating the boundary trajectory of a short-focus electron beam propagating in a free drift mode in a low-pressure ionized gas under the condition of compensation of the space charge of electrons is considered and discussed in the article. To solve the given approximation task, the root-polynomial functions of the fourth and fifth order were applied, the main features of which are the ravine character and the presence of one global minimum. As an initial approach to solving the approximation problem, the values of the polynomial coefficients are calculated by solving the interpolation problem. After this, the approximation task is solved iteratively. All necessary polynomial coefficients are calculated multiple times, taking into account the values of the function and its derivative at the reference points. The final values of polynomial coefficients of high-order root-polynomial functions are calculated using the dichotomy method. The article also provides examples of the applying fourth-order and fifth-order root-polynomial functions to approximate sets of numerical data that correspond to the description of ravine functions. The obtained theoretical results are interesting and important for the experts who study the physics of electron beams and design modern industrial electron beam technological equipment.

Keywords: approximation, interpolation, root-polynomial function, ravine function, least-square method, discrepancy, approximation error, electron beam, electron-beam technologies.

INTRODUCTION

Today, an important task regarding the further development and industrial application of electron beam technologies is the preliminary estimate of the boundary trajectory of the electron beam using different suitable approaches. Therefore, in addition to development the basic theory of electron beam optics and obtaining necessary analytical ratios and corresponded numerical methods for solving differential equations, methods of interpolation and approximation are widely used also [1–3].

A separate issue in this aspect is the evaluation of electron beam trajectories and finding the focal parameters of beams in high-voltage glow discharge (HVGD) electron guns [1; 4–7]. Main singularities of such kind of beams, at the physical point of view, is its propagation in the soft vacuum in the medium of residual gas with compensation the space charge of electrons. In additional, usually such beams are formed by the cathodes with large emission surface, therefore the convergence angle of beam is generally large and its focal diameter is not so small, range of few millimeters. Just today HVGD electron guns widely used in various branches of industry, in particular in the electronic industry, instrument building, mechanical engineering, metallurgy, automobile and aerospace industry [5–9]. The main advantages of these types of guns, regarding the possibility of their industrial application, are operation in a soft vacuum in the medium of various technological gases, including noble and active gases, high stability and reliability of operation of the HVGD electron, the relative simplicity of the design and the cheap of HVGD electron guns, as well as stability and reliability of its operation [1–3]. Ease of control the power of the electron beam both by gas dynamic lows and changing the operation pressure in discharge chamber and by the lighting of additional discharges is also possible [8; 9].

Among the advanced application of HVGD electron guns in the modern electronic production most important are follows.

1. Welding of contacts and casualization of crystals. For example, such application is very advanced in the experimental production of cryogenic low-temperature devices [10; 11].

2. Production of high-quality capacitors with the small value of current losses on the base of ceramic films [12–14].

3. Production of communication devices as receivers and transmitters of microwaves antennas on the base of high-quality ceramic films [12–14].

4. Refining of silicon ingot for obtaining the pure material for electronic industry [15–18].

Main problems of HVGD optics and energetics are well-known and have been complexly analyzed in papers [1; 19–21]. The problems of guiding short-focus electron beam in ionized gas also have been studied carefully both theoretically and experimentally, corresponded mathematical model was presented in the paper [1]. However, mathematical methods of interpolation and approximation of electron beam boundary trajectories in the medium of ionized gas still wasn't developed up to the necessary stage, corresponded mathematical function also haven't considered complexly. This shortcoming largely hinders the introduction into the industry of advanced electron-beam technologies.

In the papers [22; 23] the root-polynomial function was considered as the suitable mathematical tools to interpolation the boundary trajectories of shorty-focus electron beams in the case of its propagation in the medium of ionized gases with compensation of the space charge of electrons. Root-polynomial functions from second to fifth order and corresponded interpolation results were presented and analyzed in papers [22; 23]. The interpolation results have been compared with the accurate solution of differential equation of electron beam propagation, and corresponded interpolation error usually was smaller, than 5% [22; 23]. Therefore, the aim of investigations, which are described in this article, is forming the algorithm of approximation of boundary trajectories of short-focus electron beam, propagated in the ionized gas with compensation the space charge of electrons. Testing examples of using such approximation for the root-polynomial functions of fourth and fifth order are also considered and obtained results of numerical simulation are analyzed.

THE PREVIOUS RESEARCHES AND THEORETICAL FUNDAMENTALS OF PROPOSED APPROACH

The basic theory of polynomials interpolation and approximation is considered generally in the manual books [24; 25]. In the papers [22; 23] was considered the task of interpolation the ravine functions, which corresponded to the boundary trajectories of electron beam, propagated in the medium of ionized gas, by the root-polynomial functions, which in the general form are written as:

$$r(z)=\sqrt[n]{C_n z^n + C_{n-1} z^{n-1} + \ldots + C_1 z + C_0}\,, \tag{1}$$

where z is the longitudinal coordinate, r is the radius of the boundary trajectory of the electron beam, n is the degree of the polynomial, as well as the order of the root function, $C_0 - C_n$ are the polynomial coefficients.

The analytical relations for coefficients of forth order root-polynomial function C_0, C_1, C_2, C_3 and C_4, which, in general form, corresponding to relation (1), is written as follows [22; 23]:

$$r(z)=\sqrt[4]{C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0}\,, \tag{2}$$

are also was obtained and analyzed in the papers [22; 23].

Clear, that for 5 unknown polynomial coefficients of function (2) C_0, C_1, C_2, C_3 and C_4, with defined basic values of the spatial coordinates r_1, r_2, r_3, r_4, r_5, z_1, z_2, z_3, z_4 and z_5, corresponded set of 5 linear equation for calculation the polynomial coefficients is written as follows [22; 23]:

$$\begin{cases} C_4 z_1^4 + C_3 z_1^3 + C_2 z_1^2 + C_1 z_1 + C_0 = r_1^4; \\ C_4 z_2^4 + C_3 z_2^3 + C_2 z_2^2 + C_1 z_2 + C_0 = r_2^4; \\ C_4 z_3^4 + C_3 z_3^3 + C_2 z_3^2 + C_1 z_3 + C_0 = r_3^4; \\ C_4 z_4^4 + C_3 z_4^3 + C_2 z_4^2 + C_1 z_4 + C_0 = r_4^4; \\ C_4 z_5^4 + C_3 z_5^3 + C_2 z_5^2 + C_1 z_5 + C_0 = r_5^4. \end{cases} \tag{3}$$

For solving the set of equations (3) firstly considered the coefficients basic intermediate variables $a_{k,l}$, where k — number of iterations for solving set of equation (2) and l — number of equation i in the set (3) [22; 23]. Corresponded analytical relations are look as follows:

$$a_{1,2}=\frac{r_2^4 - r_1^4}{z_2 - z_1};\ \ a_{1,3}=\frac{r_3^4 - r_1^4}{z_3 - z_1};\ \ a_{1,4}=\frac{r_4^4 - r_1^4}{z_4 - z_1};\ \ a_{1,5}=\frac{r_5^4 - r_1^4}{z_5 - z_1};$$

$$a_{2,3}=\frac{a_{1,3} - a_{1,2}}{z_3 - z_2};\ \ a_{2,4}=\frac{a_{1,4} - a_{1,2}}{z_4 - z_2};\ \ a_{2,5}=\frac{a_{1,5} - a_{1,2}}{z_5 - z_2}. \tag{4}$$

After that, considering the second set of additional variables $b_{k,m,l}$, where parameter m is the power of variable z in the set of equations (3). Corresponded analytical relations are written as follows:

$$b_{2,3,3}=\frac{z_3^3 - z_2^3 + z_3^2 z_1 - z_2^2 z_1 + z_1^2 z_3 - z_1^2 z_2}{z_3 - z_2};$$

$$b_{2,3,3} = \frac{z_3^2 - z_2^2 + z_1 z_3 - z_1 z_2}{z_3 - z_2};$$

$$b_{2,3,3} = \frac{z_4^3 - z_2^3 + z_4^2 z_1 - z_2^2 z_1 + z_1^2 z_4 - z_1^2 z_2}{z_4 - z_2};$$

$$b_{2,2,4} = \frac{z_4^2 - z_2^2 + z_1 z_4 - z_1 z_2}{z_4 - z_2};$$

$$b_{2,3,5} = \frac{z_5^3 - z_2^3 + z_5^2 z_1 - z_2^2 z_1 + z_1^2 z_5 - z_1^2 z_2}{z_5 - z_2};$$

$$b_{2,2,5} = \frac{z_5^2 - z_2^2 + z_1 z_5 - z_1 z_2}{z_5 - z_2}; \tag{5}$$

$$b_{3,3,4} = \frac{b_{2,3,4} - b_{2,3,3}}{b_{2,2,4} - b_{2,2,3}}; \quad b_{3,3,5} = \frac{b_{2,3,5} - b_{2,3,3}}{b_{2,2,5} - b_{2,2,3}}.$$

After that, with known values of the coefficients $b_{2,2,4}$, $b_{2,2,3}$ and $b_{2,2,5}$, five additional variables from the first data set $a_{4,5}$, $a_{3,4}$, $a_{3,5}$, $a_{3,2}$, and $a_{3,1}$ as well as two new coefficients from the second set of variables $b_{3,1}$ and $b_{3,2}$, arecalculated by using such analytical relations:

$$a_{4,5} = \frac{a_{2,4} - a_{2,3}}{b_{2,2,4} - b_{2,2,3}}; \quad a_{3,4} = \frac{a_{2,4} - a_{2,3}}{b_{2,2,4} - b_{2,2,3}}; \quad a_{3,5} = \frac{a_{2,5} - a_{2,3}}{b_{2,2,5} - b_{2,2,3}}; \tag{6}$$

$$a_{3,2} = \frac{a_{2,5} - a_{2,3}}{b_{2,2,5} - b_{2,2,3}}; \quad a_{3,1} = \frac{a_{2,4} - a_{2,3}}{b_{2,2,4} - b_{2,2,3}};$$

$$b_{3,1} = \frac{b_{2,3,4} - b_{2,3,3}}{b_{2,2,4} - b_{2,2,3}}; \quad b_{3,2} = \frac{b_{2,3,5} - b_{2,3,3}}{b_{2,2,5} - b_{2,2,3}}.$$

And finally, taking into account relations (4)–(6) and the first equation of the set (3), all polynomial coefficients of the set of equation (3) are defined with applying the following relations:

$$C_4 = \frac{a_{3,2} - a_{3,1}}{b_{3,2} - b_{3,1}}; \; C_3 = \frac{a_{2,4} - a_{2,3}}{b_{2,2,4} - b_{2,2,3}} - \frac{a_{3,2} - a_{3,1}}{b_{3,2} - b_{3,1}} \cdot \frac{b_{2,3,4} - b_{2,3,3}}{b_{2,2,4} - b_{2,2,3}};$$

$$C_2 = \frac{a_{1,3} - a_{1,2}}{z_3 - z_2} - \left(\frac{a_{3,2} - a_{3,1}}{b_{3,2} - b_{3,1}} - \frac{a_{2,4} - a_{2,3}}{b_{2,2,4} - b_{2,2,3}} - \frac{a_{3,2} - a_{3,1}}{b_{3,2} - b_{3,1}} \cdot \frac{b_{2,3,4} - b_{2,3,3}}{b_{2,2,4} - b_{2,2,3}} \right) \times \tag{7}$$

$$\times \frac{z_4^3 - z_2^3 + z_4^2 z_1 - z_2^2 z_1 + z_1^2 z_4 - z_1^2 z_2}{z_4 - z_2};$$

$$C_1 = a_{1,2} - C_4 (z_2^3 + z_2^2 z_1 + z_1^2 z_2 + z_1^3) - C_3 (z_2^2 + z_1 z_2 + z_1^2) - C_2 (z_1 + z_2);$$

$$C_0 = r_1^4 - C_4 z_1^4 - C_3 z_1^3 - C_2 z_1^2 - C_1 z_1.$$

The analytical relations for coefficients of fifth order root-polynomial function C_0, C_1, C_2, C_3, C_4 and C_5, corresponding to relation (1), is written as follows [22; 23]:

$$r(z) = \sqrt[5]{C_5 z^4 + C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0}. \quad (8)$$

Therefore, the set of equation for defining the polynomial coefficient including 6 equations and generally it writing as follows [22; 23]:

$$\begin{cases} C_5 z_1^5 + C_4 z_1^4 + C_3 z_1^3 + C_2 z_1^2 + C_1 z_1 + C_0 = r_1^5; \\ C_5 z_2^5 + C_4 z_2^4 + C_3 z_2^3 + C_2 z_2^2 + C_1 z_2 + C_0 = r_2^5; \\ C_5 z_3^5 + C_4 z_3^4 + C_3 z_3^3 + C_2 z_3^2 + C_1 z_3 + C_0 = r_3^5; \\ C_5 z_4^5 + C_4 z_4^4 + C_3 z_4^3 + C_2 z_4^2 + C_1 z_4 + C_0 = r_4^5; \\ C_5 z_5^5 + C_4 z_5^4 + C_3 z_5^3 + C_2 z_5^2 + C_1 z_5 + C_0 = r_5^5; \\ C_5 z_6^5 + C_4 z_6^4 + C_3 z_6^3 + C_2 z_6^2 + C_1 z_6 + C_0 = r_6^5. \end{cases} \quad (9)$$

But the advance of proposed method of calculation the polynomial coefficients is that with using the set of coefficients for four-order function, defined by relations (4)–(6), some of that relations are also correct for defining the coefficients of fifth-order polynomial. For example, among the first set of the coefficient a only the values $a_{1,l}$ are different form relations (4), since they are including fifth order of beam radius r. Corresponded relations for defining the coefficients $a_{1,l}$ are written as follows [22; 23]:

$$a_{1,2} = \frac{r_2^5 - r_1^5}{z_2 - z_1};\ a_{1,3} = \frac{r_3^5 - r_1^5}{z_3 - z_1};\ a_{1,4} = \frac{r_4^5 - r_1^5}{z_4 - z_1};\ a_{1,5} = \frac{r_5^5 - r_1^5}{z_5 - z_1}. \quad (10)$$

Other two coefficients from the first set a_{26} and a_{36} are defined by the following analytical relations:

$$a_{2,6} = \frac{a_{1,6} - a_{1,2}}{z_6 - z_2};\quad a_{3,6} = \frac{a_{2,6} - a_{2,3}}{b_{2,2,6} - b_{2,2,3}}. \quad (11)$$

The corresponded coefficients b from the second set of additional variables are calculated for five order root-polynomial functions by analytical solving the set of linear equations (9) by the following relations:

$$b_{2,2,6} = \frac{z_6^2 - z_2^2 + z_6 z_1 - z_2 z_1}{z_6 - z_2};$$

$$b_{2,4,3} = \frac{z_3^4 - z_2^2 + z_3^3 z_1 - z_2^3 z_1 + z_3^2 z_1^2 - z_2^2 z_1^2 + z_1^3 z_3 - z_1^3 z_2}{z_3 - z_2};$$

$$b_{2,4,4} = \frac{z_4^4 - z_2^4 + z_4^3 z_1 - z_2^3 z_1 + z_4^2 z_1^2 - z_2^2 z_1^2 + z_1^3 z_4 - z_1^3 z_2}{z_4 - z_2};$$

$$b_{2,4,5} = \frac{z_5^4 - z_2^4 + z_5^3 z_1 - z_2^3 z_1 + z_5^2 z_1^2 - z_2^2 z_1^2 + z_1^3 z_5 - z_1^3 z_2}{z_5 - z_2};$$

$$b_{2,4,6} = \frac{z_6^4 - z_2^3 + z_6^2 z_1 - z_2^3 z_1 + z_1^2 z_6 - z_1^2 z_2}{z_6 - z_2}; \quad (12)$$

$$b_{2,3,6}=\frac{z_6^3-z_2^3+z_6^2z_1-z_2^2z_1+z_1^2z_6-z_1^2z_2}{z_6-z_2};$$

$$b_{3,3,4}=\frac{b_{2,4,4}-b_{2,4,3}}{b_{2,2,4}-b_{2,2,3}};\quad b_{3,4,6}=\frac{b_{2,4,6}-b_{2,4,3}}{b_{2,2,6}-b_{2,2,3}};\quad b_{3,3,6}=\frac{b_{2,3,6}-b_{2,3,3}}{b_{2,2,6}-b_{2,2,3}}.$$

With known additional variables a and b, defined by the relations (4), (10)–(12), the polynomial coefficients of root-polynomial function (8) are calculated with using following relations:

$$C_5=\frac{a_{4,6}-a_{4,5}}{b_{4,4,6}-b_{4,4,5}};\quad C_4=b_{4,4,5}\frac{a_{4,6}-a_{4,5}}{b_{4,4,6}-b_{4,4,5}}-a_{4,5};$$

$$C_3=a_{3,4}-b_{3,4,4}\frac{a_{4,6}-a_{4,5}}{b_{4,4,6}-b_{4,4,5}}-b_{3,4,4}\left(b_{4,4,5}\frac{a_{4,6}-a_{4,5}}{b_{4,4,6}-b_{4,4,5}}-a_{4,5}\right);$$

$$C_2=a_{2,3}-b_{2,4,3}\frac{a_{4,6}-a_{4,5}}{b_{4,4,6}-b_{4,4,5}}-b_{2,3,3}\left(b_{4,4,5}\frac{a_{4,6}-a_{4,5}}{b_{4,4,6}-b_{4,4,5}}-a_{4,5}\right)-$$

$$-b_{2,2,3}\left(a_{3,4}-b_{3,4,4}\frac{a_{4,6}-a_{4,5}}{b_{4,4,6}-b_{4,4,5}}-b_{3,4,4}\left(b_{4,4,5}\frac{a_{4,6}-a_{4,5}}{b_{4,4,6}-b_{4,4,5}}-a_{4,5}\right)\right);\quad (13)$$

$$C_1=a_{1,2}-b_{2,4,3}\frac{a_{4,6}-a_{4,5}}{b_{4,4,6}-b_{4,4,5}}(z_2^4+z_2^3z_1+z_2^2z_1^2+z_1^3z_2+z_1^4)-$$

$$-\left(b_{4,4,5}\frac{a_{4,6}-a_{4,5}}{b_{4,4,6}-b_{4,4,5}}-a_{4,5}\right)(z_2^4+z_2^3z_1+z_2^2z_1^2+z_1^3z_2+z_1^4)-$$

$$-C_3(z_2^2+z_2z_1+z_1^2)-C_2(z_2+z_1);$$

$$C_0=r_1^5-C_5z_1^5-C_4z_1^4-C_3z_1^3-C_2z_1^2-C_1z_1.$$

Relations (4)–(7) have been used in this work for calculation the coefficients of forth order root-polynomial function (2), and relations (10)–(13) — for calculation the corresponded coefficients of fifth order root-polynomial function (8). Such kind of ravine functions are generally characterized by one minimum, as well as by quasi-linear dependence outside the region of local minimum. In any case, such functional dependences are very suitable for approximation the trajectories of electron beam, propagated in the medium of ionized gas with compensation the space charge of electrons, because, as it was proved theoretically, the behavior of electron beams in such physical conditions is exactly the same [1; 20–23; 26–29]. An effective and simple method of calculation the optimal values of polynomial coefficients for function (3) and (8), have been used in this work for solving the task of approximation the suitable numerical data. Describing of this method, as well as corresponded examples of approximation for some of ravine functions, will be considered in the next parts of this article.

STATEMENT OF APPROXIMATION PROBLEM

In general, the approximation task is that for given approximation basis points and a given approximation function $r(z)$, for example, for function (1) with unknown coefficients C_n, C_{n-1}, ... C_1, C_0, write an analytical expression based on the method of least squares [24; 25; 30; 31].

For example, for fourth order root-polynomial function:

$$S(C_4, C_3, C_2, C_1, C_0) = \sum_{i=1}^{m} (r_i^2 - r^2(z_i, C_4, C_3, C_2, C_1, C_0))\Big|_{\min} = \tag{14}$$

$$= \sum_{i=1}^{m} \left(r_i^2 - \sqrt{C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0} \right)\Bigg|_{\min},$$

and for fifth-order function, correspondently,

$$S(C_5, C_4, C_3, C_2, C_1, C_0) = \sum_{i=1}^{m} (r_i^2 - r^2(z_i, C_5, C_4, C_3, C_2, C_1, C_0))\Big|_{\min} = \tag{15}$$

$$= \sum_{i=1}^{m} \left(r_i - \sqrt[5]{(C_5 z^5 + C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0)^2} \right)\Bigg|_{\min},$$

where n is the degree of the root-polynomial function, m is the number of reference values.

Applying known methods of solving extremal problems through the search for partial derivatives of a function of many variables [24; 25], the generalized relation (14) can be rewritten in the form of a system of algebraic differential equations as follows [24; 25; 30; 31]:

$$\begin{cases} \sum_{i=1}^{m} \left(r_i - \sqrt[4]{C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0} \right) \frac{\partial r(z_i, C_4, C_3, C_2, C_1, C_0)}{\partial C_0}; \\ \sum_{i=1}^{m} \left(r_i - \sqrt[4]{C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0} \right) \frac{\partial r(z_i, C_4, C_3, C_2, C_1, C_0)}{\partial C_1}; \\ \sum_{i=1}^{m} \left(r_i - \sqrt[4]{C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0} \right) \frac{\partial r(z_i, C_4, C_3, C_2, C_1, C_0)}{\partial C_2}; \\ \sum_{i=1}^{m} \left(r_i - \sqrt[4]{C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0} \right) \frac{\partial r(z_i, C_4, C_3, C_2, C_1, C_0)}{\partial C_3}; \\ \sum_{i=1}^{m} \left(r_i - \sqrt[4]{C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0} \right) \frac{\partial r(z_i, C_4, C_3, C_2, C_1, C_0)}{\partial C_4}. \end{cases} \tag{16}$$

Correspondently, relation (15) for fifth order root-polynomial function is re-written as follows:

$$\begin{cases} \sum_{i=1}^{m} \left(r_i - \sqrt[5]{C_5 z^5 + C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0} \right) \frac{\partial r(z_i, C_5, C_4, C_3, C_2, C_1, C_0)}{\partial C_0}; \\ \sum_{i=1}^{m} \left(r_i - \sqrt[5]{C_5 z^5 + C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0} \right) \frac{\partial r(z_i, C_5, C_4, C_3, C_2, C_1, C_0)}{\partial C_1}; \\ \sum_{i=1}^{m} \left(r_i - \sqrt[5]{C_5 z^5 + C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0} \right) \frac{\partial r(z_i, C_5, C_4, C_3, C_2, C_1, C_0)}{\partial C_2}; \\ \sum_{i=1}^{m} \left(r_i - \sqrt[5]{C_5 z^5 + C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0} \right) \frac{\partial r(z_i, C_5, C_4, C_3, C_2, C_1, C_0)}{\partial C_3}; \\ \sum_{i=1}^{m} \left(r_i - \sqrt[5]{C_5 z^5 + C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0} \right) \frac{\partial r(z_i, C_5, C_4, C_3, C_2, C_1, C_0)}{\partial C_4}. \end{cases} \tag{17}$$

The problem is that the solution of the set of equations (16), (17) in the case of a nonlinear function $r(z)$ of many variable parameters, is extremely difficult. Methods of analytical solution of some simpler approximation problems for linear, quadratic, polynomial and one-parameter functions $f(z)$, as well as for the sum of arbitrarily specified functions $\varphi_1(z)$, $\varphi_2(z)$, $\varphi_n(z)$ with unknown numerical coefficients a_0, a_1, ..., a_n are described in the textbook [30; 31], and the methods of numerical solution of systems of nonlinear equations, similar to (4), are considered in textbooks [24; 25; 32–34]. But generally, in the theory of approximation is assumed, that with increasing the number of varied variables up to 5 and more the applying methods of multicriterial analyze aren't suitable and lead to obtaining the wrong results. Usually in mathematical software tools the gradient methods, the Nelder–Mead method, the Broyden–Fletcher–Goldfarb–Shanno algorithm and others are used for solving multi-criteria optimization tasks [32–35].

Let's we will the approximation task for root-polynomial functions (2), (8) by the other approach. As an initial approximation we will choose the result of interpolation for four base points using, to calculate the polynomial coefficients of the root-polynomial function (2), (8) by applying the analytical relations (4–7; 10–13).

Regarding hat the root-polynomial function of the fourth and fifth order (2), (8) is symmetric about the axis $z = z_{\min}$, considering now the linear approximation for the second and third branches of ravine function and find the corresponding angles of inclination of the tangents $k_{\text{int}\,2}$ and $k_{\text{int}\,3}$. The solution of the linear approximation problem is simple and well-known, the corresponding analytical relations are given in textbooks [30; 31]. For the second branch of interpolation, they are written as follows:

$$r_{B2}(x) = m_{r2}^* + \frac{\sum_{i=N_{B2}}^{N_{B3}} (z_i - m_{z2}^*)(r_i - m_{r2}^*)}{\sum_{i=N_{B2}}^{N_{B3}} (z - m_{z2}^*)^2}(z - m_{z2}^*)\,; \quad m_{z2}^* = \frac{\sum_{i=N_{B2}}^{N_{B3}} z_i}{N_{B3} - N_{B2} + 1}\,;$$

$$m_{r2}^* = \frac{\sum_{i=N_{B2}}^{N_{B3}} r_i}{N_{B3} - N_{B2} + 1}; \quad k_{\text{int}\,2} = \frac{\sum_{i=N_{B2}}^{N_{B3}} (z_i - m_{z2}^*)(r_i - m_{r2}^*)}{\sum_{i=N_{B2}}^{N_{B3}} (z - m_{z2}^*)^2}\,, \tag{18}$$

and for the third branch:

$$r_{B3}(x) = m_{r3}^* + \frac{\sum_{i=N_{B3}}^{N_{End}} (z_i - m_{z3}^*)(r_i - m_{r3}^*)}{\sum_{i=N_{B3}}^{N_{End}} (z - m_{z3}^*)^2}(z - m_{z3}^*)\,; \quad m_{z3}^* = \frac{\sum_{i=N_{B3}}^{N_{End}} z_i}{N_{End} - N_{B3} + 1}\,;$$

$$m_{r3}^* = \frac{\sum_{i=N_{B3}}^{N_{End}} r_i}{N_{End} - N_{B3} + 1}; \quad k_{\text{int}\,3} = \frac{\sum_{i=N_{B3}}^{N_{End}} (z_i - m_{z3}^*)(r_i - m_{r3}^*)}{\sum_{i=N_{B3}}^{N_{End}} (z - m_{z3}^*)^2}\,, \tag{19}$$

where N_{B2} the starting point of the second approximation branch, N_{B3} is the starting point of the third approximation branch, N_{End} is the end point of the data set for approximation region.

Taking into account equations (2), (18), (19), let's we rewrite the set of equations (16) to find the minimum of the regression function (2) as follows:

$$\begin{cases} \dfrac{dr(z_1)}{dz} = k_{\text{int}\,2}; & \dfrac{dr(z_2)}{dz} = k_{\text{int}\,3}; \\ r(z_3) = r_3; & r(z_4) = r_4; \\ r(z_5) = r_5. \end{cases} \tag{20}$$

Correspondently, to fifth order root polynomial function (8), one can rewrite the set of equations (17) as follows:

$$\begin{cases} \dfrac{dr(z_1)}{dz} = k_{\text{int}\,2}; & \dfrac{dr(z_2)}{dz} = k_{\text{int}\,3}; \\ r(z_3) = r_3; & r(z_4) = r_4; \\ r(z_5) = r_5; & r(z_5) = r_5. \end{cases} \tag{21}$$

The separate problem is finding the derivations for root-polynomial functions (2), (8) in the form of suitable polynomials for providing further iterative calculations. This task was solved in provided researches with applying the tools of symbolic calculation of the MatLab scientific and technical software [32]. Corresponded obtained results for taking a derivative of the function (2) is follows:

$$\begin{aligned} &R_4(Cf_0, Cf_1, Cf_2, Cf_3, Cf_4, kf, zf) = Cf_1^4 zf^4 + 8Cf_1^3 Cf_2 zf^5 + 12Cf_1^3 Cf_3 zf^6 + \\ &+16Cf_1^3 Cf_4 zf^7 - Cf_1^3 kf^4 zf^3 + 24Cf_1^2 Cf_2^2 zf^6 + 72Cf_1^2 Cf_2 Cf_3 zf^7 + \\ &+96Cf_1^2 Cf_2 Cf_4 zf^8 - 3Cf_1^2 Cf_2 kf^4 zf^4 + 54Cf_1^2 Cf_3 zf^8 + 144Cf_1^2 Cf_3 Cf_4 zf^9 - \\ &-3Cf_1^2 Cf_3 kf^4 zf^5 + 96Cf_1^2 Cf_4^2 zf^{10} - 3Cf_1^2 Cf_4 kf^4 zf^6 + \\ &+222{,}9124 Cf_1^2 kf^4 zf^2 + 32Cf_1 Cf_2^3 zf^7 + 144Cf_1 Cf_2^2 Cf_3 zf^8 + \\ &+192Cf_1 Cf_2^2 Cf_4 zf^9 - 3Cf_1 Cf_2^2 kf^4 zf^5 + 216Cf_1 Cf_2 Cf_3^2 zf^9 + \\ &+576Cf_1 Cf_2 Cf_3 Cf_4 zf^{10} - 6Cf_1 Cf_2 Cf_3 kf^4 zf^6 + 384Cf_1 Cf_2 Cf_2^4 zf^{11} - \\ &-6Cf_1 Cf_2 Cf_4 kf^4 zf^7 + 445{,}8249 Cf_1 Cf_2 kf^4 zf^3 + 108Cf_1 Cf_3^3 zf^{10} + \\ &+432Cf_1 Cf_3^2 Cf_4 zf^{11} - 3Cf_1 Cf_3^2 kf^4 zf^7 + 576Cf_1 Cf_3 Cf_4^2 zf^{12} - \\ &-6Cf_1 Cf_3 Cf_4 kf^4 zf^8 + 445{,}8249 Cf_1 Cf_3 kf^4 zf^4 + 256Cf_1 Cf_4^3 zf^{13} - \\ &-3Cf_1 Cf_4^2 kf^4 zf^9 + 445{,}8249 Cf_1 Cf_4 kf^4 zf^5 + 1{,}6563 \cdot 10^4 Cf_1 kf^4 zf + \end{aligned} \tag{22}$$

$$+16Cf_2^4zf^8+96Cf_2^3Cf_3zf^9+128Cf_2^3Cf_4zf^{10}-Cf_2^3kf^4zf^6+$$
$$+216Cf_2^2Cf_3^2zf^{10}+576Cf_2^2Cf_3Cf_4zf^{11}-3Cf_2^2Cf_4kf^4zf^8+$$
$$+384Cf_2^2Cf_4^2zf^{12}+3Cf_2^2Cf_4kf^4zf^4+222{,}9124Cf_2^2kf^4zf^4+$$
$$+216Cf_2Cf_3^3zf^{11}+864Cf_2Cf_3^2Cf_4zf^{12}-3Cf_2Cf_3^2kf^4zf^8+$$
$$+1152Cf_2Cf_3Cf_4^2zf^{13}-6Cf_2Cf_3Cf_4kf^4zf^9+445{,}8249Cf_2Cf_3kf^4zf^5+$$
$$+512Cf_2Cf_4^3zf^{14}-3Cf_2Cf_4^2kf^4zf^{10}+445{,}8249Cf_2Cf_4kf^4zf^6+$$
$$+1{,}6563\cdot10^4Cf_2kf^4zf^2+81Cf_3^4zf^{12}+432Cf_3^3Cf_4zf^{13}+Cf_3^3kf^4zf^9+$$
$$+864Cf_3^2Cf_4^2zf^{14}-3Cf_3^2Cf_4^2kf^4zf^{10}+222{,}9124Cf_3^2kf^4zf^6+$$
$$+768Cf_3Cf_4^3zf^{15}-3Cf_3Cf_4^2kf^4zf^{11}+445{,}8249Cf_3Cf_4kf^4zf^7-$$
$$-1{,}6563\cdot10^4Cf_3kf^4zf^3+256Cf_4^4zf^{16}-Cf_4^3kf^4zf^{12}+$$
$$+222{,}9124Cf_4^2kf^4zf^8+1{,}6563\cdot10^4Cf_4kf^4zf^4+4{,}1024\cdot10^5kf^4.$$

For the derivative of fifth order root-polynomial ravine function (8) with using MatLab symbolic processor such polynomial expression have been obtained:

$$R_5(Cf_0,Cf_1,Cf_2,Cf_3,Cf_4,Cf_5,kf,zf)=Cf_1^5+10Cf_1^4Cf_2zf+15Cf_1^4Cf_3zf^2+$$
$$+20Cf_1^4Cf_4zf^3+25Cf_1^4Cf_5zf^4+40Cf_1^3Cf_2^2zf^2+120Cf_1^3Cf_2Cf_3zf^3+$$
$$+160Cf_1^3Cf_2Cf_4zf^4+200Cf_1^3Cf_2Cf_5zf^5+90Cf_1^3Cf_3^2zf^4+$$
$$+240Cf_1^3Cf_3Cf_4zf^5+300Cf_1^3Cf_3Cf_5zf^6+160Cf_1^3Cf_4^2zf^6+$$
$$+400Cf_1^3Cf_4Cf_5zf^7+250Cf_1^3Cf_5^2zf^8+80Cf_1^2Cf_2^3zf^3+$$
$$+360Cf_1^2Cf_2^3Cf_3zf^4+480Cf_1^2Cf_2^3Cf_4zf^5+600Cf_1^2Cf_2^3Cf_5zf^6+$$
$$+540Cf_1^2Cf_2^2Cf_3^2zf^5+1440Cf_1^2Cf_2Cf_3Cf_4zf^6+1800Cf_1^2Cf_2Cf_3Cf_5zf^7+$$
$$+960Cf_1^2Cf_2Cf_4^2zf^7+2400Cf_1^2Cf_2Cf_4Cf_5zf^8+1500Cf_1^2Cf_2Cf_5^2zf^9+$$
$$+270Cf_1^2Cf_3^3zf^6+1080Cf_1^2Cf_3^2Cf_4zf^7+1350Cf_1^2Cf_3^2Cf_5zf^8+$$
$$+1440Cf_1^2Cf_3Cf_4^2zf^8+3600Cf_1^2Cf_3Cf_4Cf_5zf^9+2250Cf_1^2Cf_3Cf_5^2zf^{10}+$$
$$+640Cf_1^2Cf_4^3zf^9+2400Cf_1^2Cf_4^2Cf_5zf^{10}+3000Cf_1^2Cf_4Cf_5^2zf^{11}+$$
$$+1250Cf_1^2Cf_5^3zf^{12}+80Cf_1Cf_2^4zf^4+480Cf_1Cf_2^3Cf_3zf^5+$$
$$+640Cf_1Cf_2^3Cf_4zf^6+800Cf_1Cf_2^3Cf_5zf^7+1080Cf_1Cf_2^2Cf_2^3zf^6+ \qquad (23)$$
$$+2280Cf_1Cf_2^2Cf_3Cf_4zf^7+3600Cf_1Cf_2^2Cf_3Cf_5zf^8+1920Cf_1Cf_2^2Cf_4^2zf^8+$$
$$+4800Cf_1Cf_2^2Cf_4Cf_5zf^9+3000Cf_1Cf_2^2Cf_5^2zf^{10}+1080Cf_1Cf_2Cf_3^3zf^7+$$

$$+4320Cf_1Cf_2Cf_3^2Cf_4zf^8+5400Cf_1Cf_2Cf_3^2Cf_5zf^9+5760Cf_1Cf_2Cf_3Cf_4^2zf^9+$$
$$+14400Cf_1Cf_2Cf_3Cf_4Cf_5zf^{10}+9000Cf_1Cf_2Cf_3Cf_5^2zf^{11}+$$
$$+2560Cf_1Cf_2Cf_4^3zf^{10}+9600Cf_1Cf_2Cf_4^2Cf_5zf^{11}+12000Cf_1Cf_2Cf_4Cf_5^2zf^{12}+$$
$$+5000Cf_1Cf_2Cf_4Cf_5^3zf^{13}+405Cf_1Cf_2Cf_3^4zf^8+2160Cf_1Cf_3^3Cf_4zf^9+$$
$$+2700Cf_1Cf_3^3Cf_5zf^{10}+4320Cf_1Cf_3^2Cf_4^2zf^{10}+10800Cf_1Cf_3^2Cf_4Cf_5zf^{11}+$$
$$+6750Cf_1Cf_3^2Cf_5^2zf^{12}+3840Cf_1Cf_3Cf_4^3zf^{11}+14400Cf_1Cf_3Cf_4^2Cf_5zf^{12}+$$
$$+18000Cf_1Cf_3Cf_4Cf_5^2zf^{13}+7500Cf_1Cf_3Cf_5^3zf^{14}+1280Cf_1Cf_3Cf_4^4zf^{12}+$$
$$+6400Cf_1Cf_4^3Cf_5zf^{13}+1{,}2\cdot10^4Cf_1Cf_4^2Cf_5^2zf^{14}+10^5Cf_1Cf_4Cf_5^3zf^{15}+$$
$$+3125Cf_1Cf_5^4zf^{16}-Cf_1kf^5zf+32Cf_2^5zf^5+240Cf_2^4Cf_3zf^6+$$
$$+320Cf_2^4Cf_4zf^7+400Cf_2^4Cf_5zf^8+720Cf_2^3Cf_3^2zf^7+1920Cf_2^3Cf_3Cf_4zf^8+$$
$$+2400Cf_2^3Cf_3Cf_5zf^9+1280Cf_2^3Cf_4^2zf^9+3200Cf_2^3Cf_4Cf_5zf^{10}+$$
$$+2000Cf_2^3Cf_5^2zf^{11}+1080Cf_2^2Cf_3^3zf^8+4320Cf_2^2Cf_3^2Cf_4zf^9+$$
$$+5400Cf_2^2Cf_3^2Cf_5zf^{10}+5760Cf_2^2Cf_3Cf_4zf^{10}+14400Cf_2^2Cf_3Cf_4Cf_5zf^{11}+$$
$$+9000Cf_2^2Cf_3Cf_5^2zf^{12}+2560Cf_2^2Cf_4^3zf^{11}+9600Cf_2^2Cf_4^2Cf_5zf^{12}+$$
$$+1{,}2\cdot10^4Cf_2^2Cf_4Cf_5^2zf^{13}+5\cdot10^3Cf_2^2Cf_5^3zf^{14}+810Cf_2Cf_3^4zf^9+$$
$$+4320Cf_2Cf_3^3Cf_2zf^{10}+5400Cf_2Cf_3^3Cf_5zf^{11}+8640Cf_2Cf_3^2Cf_4^2zf^{11}+$$
$$+21660Cf_2Cf_3^2Cf_4Cf_5zf^{12}+13500Cf_2Cf_3^2Cf_5^2zf^{13}+7680Cf_2Cf_3Cf_4^3zf^{12}+$$
$$+28800Cf_2Cf_3Cf_4^2Cf_5zf^{13}+36000Cf_2Cf_3Cf_4Cf_5^2zf^{14}+$$
$$+1{,}5\cdot10^4Cf_2Cf_3Cf_5^3zf^{15}+2560Cf_2Cf_4^4zf^{13}+12{,}8\cdot10^4Cf_2Cf_4^3Cf_5zf^{14}+$$
$$+2{,}4\cdot10^4Cf_2Cf_4^2Cf_5^2zf^{15}+2\cdot10^4Cf_2Cf_4Cf_5^3zf^{16}+6250Cf_2Cf_5^4zf^{17}-$$
$$-Cf_2kf^5zf^2+243Cf_3^5zf^{10}+1620Cf_3^4Cf_4zf^{11}+2025Cf_3^4Cf_5zf^{12}+$$
$$+4320Cf_3^3Cf_4^2zf^{12}+10800Cf_3^3Cf_4Cf_5zf^{13}+6750Cf_3^3Cf_5^2zf^{14}+$$
$$+5760Cf_3^2Cf_4^3zf^{13}+2{,}16\cdot10^4Cf_3^2Cf_4^2Cf_5zf^{14}+2{,}7\cdot10^4Cf_3^2Cf_4Cf_5zf^{15}+$$
$$+11250Cf_3^2Cf_5^3zf^{16}+3840Cf_3Cf_4^4zf^{14}+1{,}92\cdot10^4Cf_3Cf_4^3Cf_5zf^{15}+$$
$$+3{,}6\cdot10^4Cf_3Cf_4^2Cf_5^2zf^{16}+3\cdot10^4Cf_3Cf_4Cf_5^3zf^{17}+9375Cf_3Cf_5^4zf^{18}-$$
$$-Cf_3kf^5zf^3+1024Cf_4^5zf^{15}+6400Cf_4^4Cf_5zf^{16}+1{,}6\cdot10^4Cf_4^3Cf_5^2zf^{17}+$$
$$+2\cdot10^4Cf_4^2Cf_5^3zf^{18}+12{,}5\cdot10^4Cf_4Cf_5^4zf^{19}-Cf_4kf^5zf^4+$$
$$+3125Cf_5^5zf^{20}-Cf_5kf^5zf^5-Cf_0kf^5.$$

Obtained polynomial relations (22), (23) have been applied in provided investigation for iterative calculation the values of polynomial coefficient with using the method of consistent upper relaxation [33–36]. These relations included the values of the coefficients root-polynomial function (2), (8), as well as zf coordinate and slope angles of ravine function kf. The particularities of proposed algorithm, as well as some examples of solving approximation task, will be considered in the next sections of the article.

PARTICULARITIES OF DEVELOPED ITERATIVE ALGORITHM FOR SOLVING THE APPROXIMATION TASK

Iterative algorithm for approximation the ravine sets of numerical data by using root-polynomial dependences four and fifth order (2), (8) generally including the follow necessary steps.

1. As basic approach the interpolation task is solved and the basic values of polynomial coefficients are calculated withusing relations (4)–(7) for fourth order function (2), and by the relations (10)–(13) for fifth order function (8).

2. Solving the relaxation task for the finding values of polynomial coefficient with using iterative algorithm. Corresponded iterative relations for function (2), taking into account (22), are the follows:

$$\begin{aligned} C_1^i &= -R_4(C_0^{i-1}, C_1^{i-1}, C_2^{i-1}, C_3^{i-1}, C_4^{i-1}, kf_1, zf_3) w_{C_1}; \\ C_3^i &= R_4(C_0^{i-1}, C_1^i, C_2^{i-1}, C_3^{i-1}, C_4^{i-1}, kf_2, zf_4) w_{C_3}; \\ C_4^i &= ((r_2^4 - C_3^i zf_2^3 - C_2^{i-1} zf_2^2 - C_1^i zf_2 - C_0^{i-1}) w_{C_4}) \big/ zf_2^4; \\ C_2^i &= ((r_1^4 - C_4^i zf_1^4 - C_2^{i-1} zf_1^2 - C_1^i zf_1 - C_0^{i-1}) w_{C_2}) \big/ zf_1^2; \\ C_0^i &= (r_1^4 - C_4^i zf_1^4 - C_2^i zf_1^2 - C_1^i zf_1 - C_0^{i-1}) w_{C_0}. \end{aligned} \tag{24}$$

For fifth order root-polynomial function (8), taking into account (23), the iterative relations, in the general form, are rewritten as follows:

$$\begin{aligned} C_1^i &= -R_5(C_0^{i-1}, C_1^{i-1}, C_2^{i-1}, C_3^{i-1}, C_4^{i-1}, C_5^{i-1}, kf_1, zf_3) w_{C_1}; \\ C_3^i &= R_5(C_0^{i-1}, C_1^i, C_2^{i-1}, C_3^{i-1}, C_4^{i-1}, C_5^{i-1}, kf_2, zf_4) w_{C_3}; \\ C_5^i &= ((r_6^5 - C_4^{i-1} zf_6^4 - C_3^i zf_6^3 - C_2^{i-1} zf_6^2 - C_1^i zf_6 - C_0^{i-1}) w_{C_5}) \big/ zf_6^5; \\ C_4^i &= ((r_2^5 - C_5^i zf_2^5 - C_3^i zf_2^3 - C_2^{i-1} zf_2^2 - C_1^i zf_6 - C_0^{i-1}) w_{C_4}) \big/ zf_2^4; \\ C_2^i &= ((r_1^5 - C_5^i zf_1^5 - C_4^i zf_1^4 - C_3^i zf_1^3 - C_1^i zf_1 - C_0^{i-1}) w_{C_2}) \big/ zf_1^2; \\ C_0^i &= (r_5^5 - C_5^i zf_5^5 - C_4^i zf_5^4 - C_3^i zf_5^3 - C_2^i zf_5^3 - C_1^i zf_5) w_{C_0}. \end{aligned} \tag{25}$$

3. Finding the best function of approximation by calculation the error with using relations (14), (15) for least-square method.

4. Finding the new optimal values of polynomial coefficients wit using dichotomy calculations, namely:

$$C_l^i = \frac{C_l^{i-1} - C_l^{i-2}}{2}, \qquad (26)$$

where l — the number of polynomial coefficient, defined in the basic relations (4)–(6), (10)–(13).

As shown the tests calculation experiments, the convergence of proposed algorithm is provided by 4–6 iterations. It also must be taking into account, that first iteration for high order root-polynomial function, which have been obtained by interpolation through $n + 1$ points, where n is the polynomial order, is really usually close to optimal. The best solution has been chosen by the smallest value of sum in relation (14), (15), therefore the well-known least-square method is considered in this research as the criterium of optimization task [30–39]. Choosing of basic points is also very important, therefore considering the different sets of its for finding the best approximative root-polynomial function is also have been provided. Analyzing the interpolation error of the different order root-polynomial functions in dependence on choosing the set of basic points have been provided generally in the papers [22; 23].

And finally, choose the correct values of $w_{C_0} - w_{C_5}$ relaxation coefficient is very important problem for providing the stability of convergence of proposed iteration algorithm, because the value problem of coefficients $C_0 - C_5$ in the relations (24)–(26) are usually, for the practice engineering tasks, can be both extra low or extra high.

Some examples of solving approximation for the ravine data sets with using root-polynomial functions (2), (8) will be considered in the next part of the article.

SOME TESTING EXAMPLES OF SOLVING THE APPROXIMATION TASK

Example 1. Find the coefficients of fourth order approximative root polynomial function for ravine function data set with one global minimum. Corresponded digital data are presented at Table 1.

Table 1. The first set of numerical data of the ravine function, have been used for testing the software tools for solving the approximation problem

z, mm	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3
r, mm	2.5	2.4	2.3	2.1	1.8	1.75	2.11	2.3	2.4

The obtained graphic results of solving the approximation task of this example with using relations (24), (26) are presented at Fig 1.

Corresponded values of relaxation coefficient for providing the convergence of iterative relations (24) have been defined as follows:

$$w_{C_1} = -1.7995 \cdot 10^6; \; w_{C_3} = 1.9163 \cdot 10^{-5}; \; w_{C_4} = 1; \; w_{C_2} = -1.131; \; w_{C_0} = 0.945.$$

The values of calculated polynomial coefficients are presented at Table 2. The error of approximation δ, defined by the equation (14) as sum of squares of differences between basic points and value4 of approximative function (2), also noted in this table. The relative error of approximation is calculated as $\delta[\%] = r_{\max}^2 (S/()) \cdot 100$, де $r_{\max}$ — the maximum value of electron beam radius.

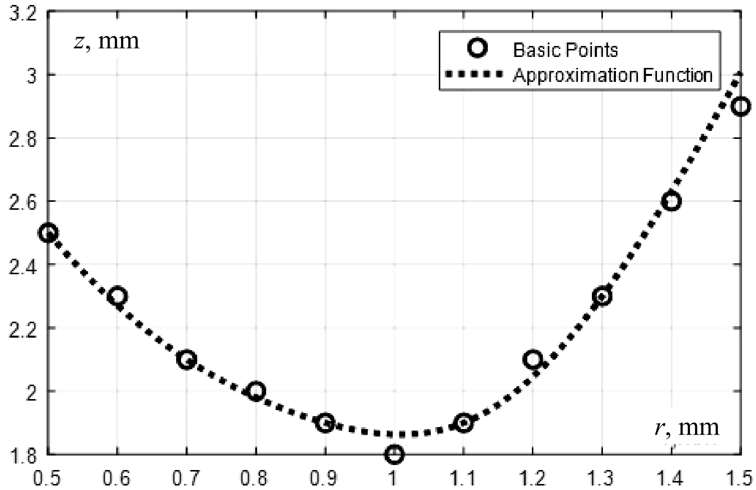

Fig. 1. The results of solving the approximation task with using fourth order root-polynomial function (2) for data set, presented at Table 1. Corresponded values of polynomial coefficients and the approximation error are given at Table 2

Table 2. The results of solving the approximation task for numerical data, presented in the Table 1

Nubmer of iteration	**Values of polynomial coefficients**					**δ, %**
	C_0	C_1	C_2	C_3	C_4	
Basic Approach	–493.16	2750.4	4976.8	3690.6	–961.7	8
First iteration	–493.179	2750.41	4976.81	3686.9	–956.2	5
Second iteration	–493.172	2750.408	4976.81	3688.7	–958.961	4
Third iteration	–493.17	2750.408	4976.7	3689.68	–960.33	3

In this example first iteration by the polynomial coefficients have been realized with applying relations (24), (25), and the second and third iterations, in which have been obtained the best solution of approximation task, have been provided with using relation (26).

Example 2. Find the coefficients of fourth order approximative root-polynomial function for ravine function data set with one global minimum. Corresponded digital data are presented in Table 3.

Table 3. The second set of numerical data of the ravine function, have been used for testing the software tools for solving the approximation problem

z, mm	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
r, mm	2.5	2.3	2.1	2.0	1.9	1.8	1.9	2.1	2.3	2.6	2.9

The obtained graphic results of solving the approximation task of this example with using one iteration step are presented at Fig 2. Really the task of interpolation by choose 5 basis points among the 11 given have been solved in this case, and the interpolation error was smaller, than 2%. The calculated values of polynomial coefficients for this test example are presented in Table 4.

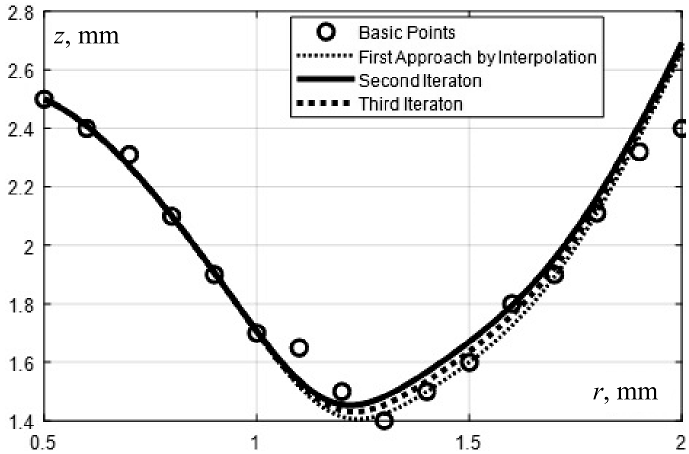

Fig. 2. The results of solving the approximation task with using fourth order root-polynomial function (2) for data set, presented at Table 3. Corresponded values of polynomial coefficients and the approximation error are given at Table 4

Table 4. The results of solving the approximation task for numerical data, presented in the Table 3

Nubmer of iteration	Values of polynomial coefficients					δ, %
	C_0	C_1	C_2	C_3	C_4	
Basic Approach	328.6	–1251.26	1953.66	–1417.8	398.9	1.5

Example 3. Find the coefficients of fifth order approximative root polynomial function for ravine function data set with one global minimum. Corresponded digital data are presented at Table 5.

Table 5. The third set of numerical data of the ravine function, have been used for testing the software tools for solving the approximation problem

z, mm	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
r, mm	2.5	2.4	2.31	2.1	1.9	1.7	1.65	1.63
z, mm	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
r, mm	1.6	1.55	1.7	1.8	1.9	2.11	2.32	2.4

In this and in the next example the corresponded values of relaxation coefficient for providing the consvergence of iterative relations (25) have been defined as follows:

$$w_{C_1} = -5{,}53 \cdot 10^{-7};\ w_{C_3} = 9{,}5 \cdot 10^{-7};\ w_{C_5} = 1;\ w_{C_4} = 1;\ w_{C_2} = 0{,}39;\ w_{C_0} = 0{,}145.$$

The obtained graphic results of solving the approximation task of this example with using one iteration step are presented at Fig 3. It is clear from this example, that considered type of the root-polynomial functions is not very suitable for approximation the ravine data sets with large area of minimum numerical values. Corresponded approximation error was greater than 12 %. But for the left and right branches of considered data set the error of approximation is much smaller, nearly few percents.

Example 4. Find the coefficients of fifth order approximative root polynomial function for ravine function data set with one global minimum. Corresponded digital data are presented at Table 6.

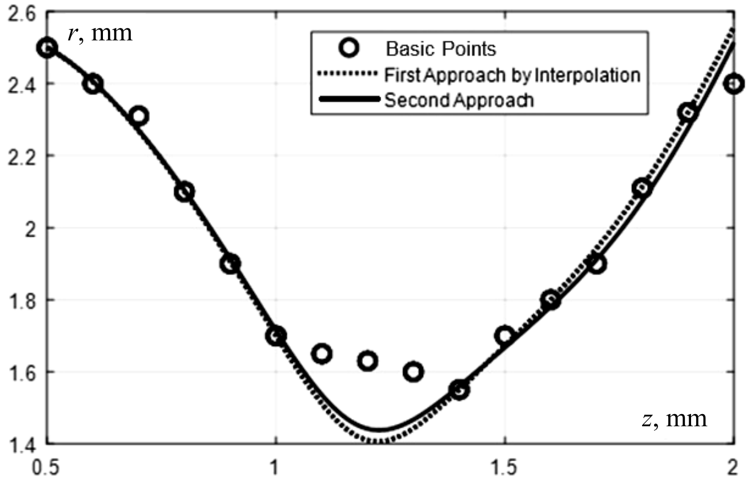

Fig. 3. The results of solving the approximation task with using fifth order root-polynomial function (8) for data set, presented at Table 5. Corresponded approximation error for the left and right branches of considered ravine function was smaller, than few percent

Table 6. The fourth set of numerical data of the ravine function, have been used for testing the software tools for solving the approximation problem

z, mm	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
r, mm	2.5	2.4	2.31	2.1	1.9	1.7	1.65	1.5
z, mm	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
r, mm	1.4	1.5	1.6	1.8	1.9	2.11	2.32	2.4

The obtained graphic results of solving the approximation task of this example with using 3 iteration step are presented at Fig 4.

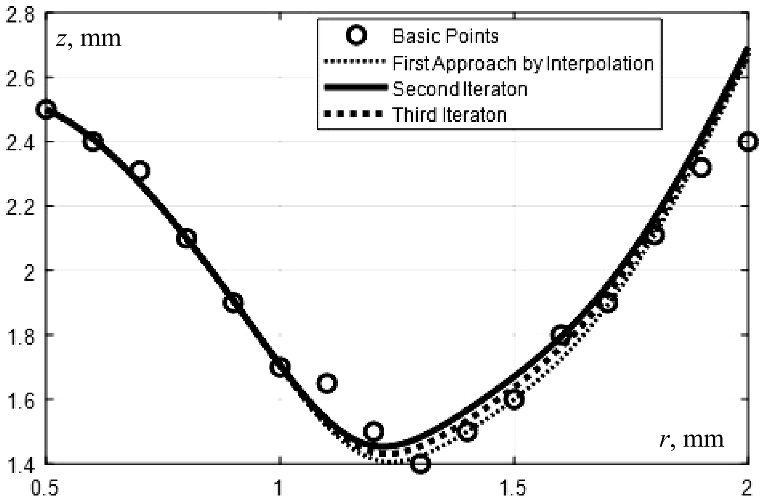

Fig. 4. The results of solving the approximation task with using fifth order root-polynomial function (8) for data set, presented at Table 6. Corresponded values of the polynomial coefficients and approximation error are given in Table 7

Table 7. The results of solving the approximation task for numerical data, presented in the Table 6

Nubmer of iteration	Values of polynomial coefficients						δ, %
	C_0	C_1	C_2	C_3	C_4	C_5	
Basic Approach	–231.51	2340.2	–5259.9	5083.15	–2278.7	392.01	5
First iteration	–39.32	2340.1	–5260.0	5082.7	–2276.1	392.1	4
Second iteration	–151.0	2340.0	–5259.9	5083.0	–2278.2	392.1	3

In this example, the iterative process was carried out in the same way as in example 1. The first values of polynomial coefficients were obtained by solving interpolation task. Therefore, for defining the coefficients of the root-polynomial function (8) relations (10)–(13) was applied. In this step of solving approximation task 5 basis points have been choose among the 16 given. By the such way the basic the approach for calculation the polynomial coefficients have been provided. At the second iteration the values of polynomial coefficient have been calculated iteratively with using relations (25), and in the third iteration — with using relation (26). Corresponded results of for the coefficients of fifth order root-polynomial function, as well as the estimated value of approximation error, have been presented in the Table 7.

ANALYZING OF OBTAINED RESULTS AND ITS' DISCUSSION

The provided numerical experiments for solving the approximation task for ravine data set with using high order root-polynomial functions (2), (8) shown, that generally by providing simple iterative process with relaxation by using relations (24), (25) is really possible to find the optimal correct values of polynomial coefficient. Estimated theoretically approximation error in most cases was in range of few percent, and, taking into account, that usually approximated experimental data for the trajectories of electron beams are included the large amount instrumental errors, such estimations, in the most cases, are generally useful from the practical point of view.

The main distinguishing feature of proposed algorithm is possibilities of obtaining the correct results for approximation task just on the first step of calculations by solving the simpler interpolation task. Such approach is very effective on the practical point of view for developing corresponded software for simulation the boundary trajectories of electron beams.

The provided testing experiment also shown, that root-polynomial functions of high order, like (8), (12), are generally not suitable for approximation the ravine data sets with the large area near the minimum, but such kind of functions isn't corresponded to the trajectories of electron beam in standard physical conditions. Usually, the region of beam focus is relatively small, and, as shown the provided numerical experiments, such data sets can be approximated by the high order root-polynomial functions with small error.

The provided theoretical analyze also shown, that for obtaining the convergence of proposed iterative approximation algorithm correct choosing of relaxation coefficients $w_{C_0} - w_{C_5}$ is very important numerical factor.

Analyzing of another advanced possibilities of applying high order root-polynomial functions for accurate approximation of ravine data sets and forming the corresponded algorithms is the subject of further theoretical researches.

But, generally, the complicity of proposed iterative algorithm is connected only with solving the strong non-linear task for derivatives of ravine functions (2), (8) and to numerical solving the complex polynomial relations (22), (23). The provided numerical experiments shown, that the polynomial coefficient, which are calculated throw the derivatives by numerical solving the relations (22), (23), are strongly depended on relaxation parameters and the values of these parameters are magnificently grater, than for the polynomial coefficients, which are calculated independently by using simple relations (2), (8). But it is also clear from the provided testing numerical experiments, that the approximation error is mostly defined by the polynomial coefficients, which values are calculated through the derivations. And the values of coefficients, calculated directly by the relations (2), (8), are generally stable and corresponded relaxation parameter is in the range of medium value, it is not very large and not so small. But since usually ravine functions in the region of a local minimum are non-linear, it is often complex non-linear equations for derivatives, like (22), (23), are solved. Its makes possible to ensure, that the smallest error and the optimal values of the polynomial coefficients for solving the approximation problem have been choose.

Choosing the optimal values of relaxation parameters for calculation the polynomial coefficients throw the derivatives is also the subject of further theoretical researches.

CONCLUSION

The theoretical researches, have been provided and described in this work, as well as realized test numerical experiments, given in the corresponded examples, have shown, that applying of high order root-polynomial functions, which have been determinate by the analytical relations (2), (8), always leads to solving the task of approximation the numerical sets of ravine functions data with the small value of error. Therefore, such functions can be used successfully for approximation the boundary trajectories of electron beams, propagated in the ionized gas with compensation the space charge of beam electrons. Such approximation can also be applied to the noisy experimental data, which included experimental errors. Therefore, studied and proposed in these researches the theoretical approach and the numerical algorithm of the approximation of digital data are very important from the practical point of view for further development of modern industrial electron beam equipment with applying HVGD electron gun as the source of intensive beams. For solving this task proposed iterative algorithm has to be integrated with the software tools for treatment of experimental photographs, where the beam propagates through ionized gas. Analyzing of the brightness of a burning discharge on such photographs gives the important experimental information about the boundary trajectories of propagated electron beam in the real physical conditions. [1].

Advanced singularities of proposed iterative algorithm, including the possibilities of approximation the ravine data sets with a large area of the global minimum, as well as studying of influence of the derivative of root-polynomial functions on the values of relaxation coefficients, are the subject of further theoretical researches. But, in any case, the results of provided theoretical researches, presented in this article, are enough for estimation the boundary trajectories of electron beams, propagated in ionizing gas. Therefore, theoretical results, which have already been obtained, are very interesting and important for experts in the branch of elaboration of modern electron-beam equipment and its industrial application.

REFERENCES

1. I. Melnyk, S. Tuhai, and A. Pochynok, "Universal Complex Model for Estimation the Beam Current Density of High Voltage Glow Discharge Electron Guns," *Lecture Notes in Networks and Systems*, vol. 152, pp. 319–341, 2021. Available: https://www.springer.com/gp/book/9783030583583
2. S.V. Denbnovetsky, V.G. Melnyk, and I.V. Melnyk, "High voltage glow discharge electron sources and possibilities of its application in industry for realising of different technological operations," *IEEE Transactions on Plasma Science*, vol. 31, no. 5, pp. 987–993, 2003. Available: https://ieeexplore.ieee.org/document/1240048
3. S. Denbnovetskiy et al., "Principles of operation of high voltage glow discharge electron guns and particularities of its technological application," *Proceedings of SPIE, The International Society of Optical Engineering*, pp. 10445–10455, 2017. Available: https://www.spiedigitallibrary.org/conference-proceedings-of-spie/10445/104455R/Principles-of-operation-of-high-voltage-glow-discharge-electron-guns/10.1117/12.2280736.short
4. M. Grechanyuk, A. Melnyk, I. Grechanyuk, V. Melnyk, and D. Kovalchuk, "Modern electron beam technologies and equipment for melting and physical vapor deposition of different materials," *Electrotechnics and Electronics* (E+E), vol. 49, no. 5–6, pp. 115–121, 2014.
5. T. Hassel, N. Beniyash, N. Murray, R. Konya, and Fr.-W. Bach, "Non-vacuum electron beam for cutting application," *Electrotechnics and Electronics (E+E)*, vol. 47, no. 5–6, pp. 146–151, 2012.
6. V. Vassilieva, K. Vutova, and V. Donchev, "Recycling of alloy steel by electron beam melting," *Electrotechnics and Electronics (E+E)*, vol. 47, no. 5–6, pp. 142–145, 2012.
7. G. Mattausch et al., "Gas discharge electron sources – proven and novel tools for thin-film technologies," *Electrotechnics and Electronics (E+E)*, vol. 49, no. 5–6, pp. 183–195, 2014.
8. S.V. Denbnovetsky, V.I. Melnyk, I.V. Melnyk, and B.A. Tugay, "Model of control of glow discharge electron gun current for microelectronics production applications," *Proceedings of SPIE. Sixth International Conference on "Material Science and Material Properties for Infrared Optoelectronics"*, vol. 5065, pp. 64–76, 2003. Available: https://www.spiedigitallibrary.org/conference-proceedings-of-spie/5065/0000/ Model-of-control-of-glow-discharge-electron-gun-current-for/10.1117/12.502174.short

9. I.V. Melnyk, “Estimating of current rise time of glow discharge in triode electrode system in case of control pulsing,” *Radioelectronic and Communication Systems*, vol. 56, no. 12, pp. 51–61, 2017. Available: http://radioelektronika.org/article/view/S0735272713120066
10. A.A. Druzhinin, I.P. Ostrovskii, Y.N. Khoverko, N.S. Liakh-Kaguy, and A.M. Vuytsyk, “Low temperature characteristics of germanium whiskers,” *Functional Materials*, no. 21(2), pp. 130–136, 2014. Available: https://nanoscalereslett.springeropen.com/articles/10.1186/s11671-017-1923-1
11. A.A. Druzhinin, I.A. Bolshakova, I.P. Ostrovskii, Y.N. Khoverko, and N.S. Liakh-Kaguy, “Low temperature magnetoresistance of InSb whiskers,” *Materials Science in Semiconductor Processing*, no. 40, pp. 550–555, 2015. Available: https://academic-accelerator.com/search?Journal=Druzhinin
12. A. Zakharov, S. Rozenko, S. Litvintsev, and M. Ilchenko, “Trisection Bandpass Filter with Mixed Cross-Coupling and Different Paths for Signal Propagation,” *IEEE Microwave Wireless Component Letters*, vol. 30, no. 1, pp. 12–15, 2020.
13. A. Zakharov, S. Litvintsev, and M. Ilchenko, “Trisection Bandpass Filters with All Mixed Couplings,” *IEEE Microwave Wireless Components Letter*, vol. 29, no. 9, pp. 592–594, 2019. Available: https://ieeexplore.ieee.org/abstract/document/8782802
14. A. Zakharov, S. Rozenko, and M. Ilchenko, “Varactor-tuned microstrip bandpass filter with loop hairpin and combline resonators,” *IEEE Transactions on Circuits Systems. II. Experimental Briefs*, vol. 66, no. 6, pp. 953–957, 2019. Available: https://ieeexplore.ieee.org/document/8477112
15. T. Kemmotsu, T. Nagai, and M. Maeda, “Removal Rate of Phosphorous form Melting Silicon,” *High Temperature Materials and Processes*, vol. 30, no. 1–2, pp. 17–22, 2011. Available: https://www.degruyter.com/journal/key/htmp/30/1-2/html
16. J.C.S. Pires, A.F.B. Barga, and P.R. May, “The purification of metallurgically grade silicon by electron beam melting,” *Journal of Materials Processing Technology*, vol. 169, no. 1, pp. 347–355, 2005. Available: https://www.academia.edu/9442020/The_purification_of_metallurgical_grade_silicon_by_electron_beam_melting
17. D. Luo, N. Liu, Y. Lu, G. Zhang, and T. Li, “Removal of impurities from metallurgically grade silicon by electron beam melting,” *Journal of Semiconductors*, vol. 32, no. 3, Article ID 033003, 2011. Available: http://www.jos.ac.cn/en/article/doi/10.1088/ 1674-4926/ 32/3/033003
18. D. Jiang, Y. Tan, S. Shi, W. Dong, Z. Gu, and R. Zou, “Removal of phosphorous in molten silicon by electron beam candle melting,” *Materials Letters*, vol. 78, pp. 4–7, 2012.
19. I.V. Melnyk, “Numerical simulation of distribution of electric field and particle trajectories in electron sources based on high-voltage glow discharge,” *Radioelectronic and Communication Systems*, vol. 48, no. 6, pp. 61–71, 2005. Available: http://radioelektronika.org/article/view/S0735272705060087
20. S.V. Denbnovetsky, J. Felba, V.I. Melnik, and I.V. Melnik, “Model of Beam Formation in a Glow Discharge Electron Gun With a Cold Cathode,” *Applied Surface Science*, vol. 111, pp. 288–294, 1997. Available: https://www.sciencedirect.com/ science/article/pii/S0169433296007611?via%3Dihub
21. J.I. Etcheverry, N. Mingolo, J.J. Rocca, and O.E. Martınez, “A Simple Model of a Glow Discharge Electron Beam for Materials Processing,” *IEEE Transactions on Plasma Science*, vol. 25, no. 3, pp. 427–432, 1997.
22. I. Melnyk, S. Tuhai, and A. Pochynok, “Interpolation of the Boundary Trajectories of Electron Beams by the Roots from Polynomic Functions of Corresponded Order,” *2020 IEEE 40th International Conference on Electronics and Nanotechnology (ELNANO). Conference Proceedings*, pp. 28–33. Available: https://ieeexplore.ieee.org/ servlet/opac?punumber=9085228
23. I. Melnik, S. Tugay, and A. Pochynok, “Interpolation Functions for Describing the Boundary Trajectories of Electron Beams Propagated in Ionised Gas,” *15-th Interna-*

tional Conference on Advanced Trends in Radioelectronics, Telecommunications and Computer Engineering (TCSET – 2020) Conference Proceedings, pp. 79–83, 2020. Available: https://ieeexplore.ieee.org/document/9088609
24. G.M. Phillips, *Interpolation and Approximation by Polynomials*. Springer, 2023, 312 p. Available: http://bayanbox.ir/view/2518803974255898294/George-M.-Phillips-Interpolation-and-Approximation-by-Polynomials-Springer-2003.pdf
25. N. Draper and H. Smith, *Applied Regression Analysis*; 3rd Edition. Wiley Series, 1998, 706 p. Available: https://www.wiley.com/en-us/Applied+Regression+ Analysis,+3rd+Edition-p-9780471170822
26. J.D. Lawson, *The Physics of Charged-Particle Beams*. Oxford: Clarendon Press, 1977, 446 p. Available: https://www.semanticscholar.org/paper/The-Physics-of-Charged-Particle-Beams-Stringer/80b5ee5289d5efd8f480b516ec4bade0aa529ea6
27. M. Reiser, *Theory and Design of Charged Particle Beams*. John Wiley & Sons, 2008, 634 p. Available: https://www.wiley.com/en-us/Theory+and+Design+of+Charged +Particle +Beams-p-9783527617630
28. M. Szilagyi, *Electron and Ion Optics*. Springer Science & Business Media, 2015, 539 p. Available: https://www.amazon.com/Electron-Optics-Microdevices-Miklos-Szilagyi/dp/1461282470
29. S.J.R. Humphries, *Charged Particle Beams*. Courier Corporation, 2013, 834 p. Available: https://library.uoh.edu.iq/admin/ebooks/76728-charged-particle-beams---s.-humphries.pdf
30. E. Wentzel and L. Ovcharov, *Applied Problems of Probability Theory*. Mir, 1998, 432 p. Available: https://mirtitles.org/2022/06/03/applied-problems-in-probability-theory-wentzel-ovcharov/
31. J.A. Gubner, *Probability and random processes for electrical and computer engineers*. Cambridge, UK: Cambridge University Press, 2006. Available: http://www.cambridge.org/gb/academic/subjects/engineering/communications-and-signal-processing/probability-and-random-processes-electrical-and-computer-engineers
32. J.H. Mathews and K.D. Fink, *Numerical Methods Using MATLAB*; 3rd Edition. Amazon, 1998, 720 p. Available: https://www.abebooks.com/book-search/title/numerical-methods-using-matlab/author/john-mathews-kurtis-fink/?cm_mmc=ggl-_-COMUS_ETA_DSA-_-naa-_-naa&gclid=CjwKCAiAh9qdBhAOEiwAvxIok6hZ7XHTvi420qugGwqNZ20QF4PyaaJai-74Z0EK2c3dbVRqo1P17hoCP2wQAvD_BwE
33. C. Mohan and K. Deep, *Optimization Techniques*. New Age Science, 2009, 628 p. Available: https://www.amazon.com/Optimization-Techniques-C-Mohan/dp/ 1906574219
34. M.K. Jain, S.R.K. Iengar, and R.K. Jain, *Numerical Methods for Scientific & Engineering Computation*. New Age International Pvt. Ltd., 2010, 733 p. Available: https://www.google.com.ua/url?sa=t&rct=j&q=&esrc=s&source=web&cd=&ved=2ahUKEwippcuT7rX8AhUhlYsKHRfBCG0QFnoECEsQAQ&url=https%3A%2F%2Fwww.researchgate.net%2Fprofile%2FAbiodun_Opanuga%2Fpost%2Fhow_can_solve_a_non_linear_PDE_using_numerical_method%2Fattachment%2F59d61f7279197b807797de30%2FAS%253A284742038638596%2540144489920043%2Fdownload%2FNumerical%2BMethods.pdf&usg=AOvVaw0MjNl3K877lVWUWw-FPwmV
35. S.C. Chapra and R.P. Canale, *Numerical Methods for Engineers*; 7th Edition. McGraw Hill, 2014, 992 p. Available: https://www.amazon.com/Numerical-Methods-Engineers-Steven-Chapra/dp/007339792X
36. I.N. Bronshtein, K.A. Semendyayev, G. Musiol, and H. Mühlig, *Handbook of Mathematics*; 5th Edition. Springer, 2007, 1164 p.

Received 02.02.2023

INFORMATION ON THE ARTICLE

Igor V. Melnyk, ORCID: 0000-0003-0220-0615, National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Ukraine, e-mail: imelnik@phbme.kpi.ua

Alina V. Pochynok, ORCID: 0000-0001-9531-7593, Research Institute of Electronics and Microsystem Technology of the National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Ukraine, e-mail: alina_pochynok@yahoo.com

БАЗОВИЙ АЛГОРИТМ АПРОКСИМАЦІЇ ГРАНИЧНОЇ ТРАЄКТОРІЇ КОРОТКОФОКУСНОГО ЕЛЕКТРОННОГО ПУЧКА ЗА ДОПОМОГОЮ КОРЕНЕВО-ПОЛІНОМІАЛЬНИХ ФУНКЦІЙ ЧЕТВЕРТОГО ТА П'ЯТОГО ПОРЯДКІВ / І.В. Мельник, А.В. Починок

Анотація. Розглянуто новий ітераційний метод апроксимації граничної траєкторії короткофокусного електронного пучка, який поширюється в режимі вільного дрейфу в іонізованому газі низького тиску за умови компенсації просторового заряду електронів. Використано кореневo-поліноміальні функції четвертого та п'ятого порядків, головними особливостями яких є яружний характер та наявність одного глобального мінімуму. Як початкове наближення для розв'язування апроксимаційної задачі розраховано значення поліноміальних коефіцієнтів через розв'язання задачі інтерполяції. Задачу апроксимації розв'язано ітераційно. Для цього поліноміальні коефіцієнти обчислено багаторазово з урахуванням значень функції та її похідної у відлікових точках. Остаточні значення поліноміальних коефіцієнтів кореневo-поліноміальних функцій високого порядку розраховано з використанням методу дихотомії. Наведено приклади використання кореневo-поліноміальних функцій четвертого та п'ятого порядків для апроксимації наборів числових даних, які відповідають опису яружних функцій. Отримані теоретичні результати є цікавими та корисними для спеціалістів, які вивчають фізику електронних пучків та займаються проектуванням сучасного промислового електронно-променевого технологічного обладнання.

Ключові слова: апроксимація, інтерполяція, кореневo-поліноміальна функція, яружна функція, метод найменших квадратів, нев'язка, похибка апроксимації, електронний пучок, електронно-променеві технології.