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GENERALIZED SCENARIOS OF TRANSITION TO CHAOS IN IDEAL DYNAMIC SYSTEMS

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Abstract. The implementation of a new scenario of transition to chaos in the classical Lorenz system has been discovered. Signs of the presence of an implementation of the generalized intermittency scenario for dynamic systems are described. Phase-parametric characteristics, Lyapunov characteristic exponents, distributions of invariant measures, and Poincaré sections are constructed and analyzed in detail, which confirm the implementation of the generalized intermittency scenario in an ideal Lorenz system.

Keywords: ideal dynamic system, regular and chaotic attractors, generalized intermittency scenario.

ITRODUCTION

The study of the ways in which deterministic chaos arises in dynamic systems leads to the necessity to study scenarios of transition to chaos. The scenario of transition to chaos is understood as a sequence of bifurcations, as a result of which a regular steady state regime, for example a periodic one, is replaced by chaotic steady state regime. If the dynamic system is dissipative, then such steady-state regimes will be attractors of system. Despite the huge variety of dynamic systems, there are two types of scenarios for the transition from regular to chaotic modes, which are implemented in almost all dynamic systems. This is the Feigenbaum scenario, during which there is a transition from limit cycle to chaotic attractor through an infinite cascade of bifurcations of period doubling of limit cycle [1; 2]. The second typical scenario is Manneville–Pomeau intermittency [3–5]. When this scenario is implemented, the transition from limit cycle to chaotic attractor occurs in one hard bifurcation. As a result of such bifurcation, the limit cycle disappears and a chaotic attractor appears in the system. Moreover, movement along the trajectories of a chaotic attractor consists of two unpredictably alternating phases — laminar and turbulent.

Recently, a number of new scenarios for transitions to chaos have been described that generalize Manneville–Pomeau and Feigenbaum scenarios. An overview of these scenarios is given in [6]. One of these new scenarios is the so-called generalized intermittency [7; 8]. In contrast to classical intermittency, this scenario describes the transition from a chaotic (hyper-chaotic) attractor of one type to a chaotic (hyper-chaotic) attractor of another type. The implementation of the generalized intermittency scenario was discovered in a number of hydrodynamic, electro-elastic and pendulum systems [7–12], both for traditional attractors and for atypical maximal attractors.

FORMULATION OF THE PROBLEM

All implementations of the generalized intermittency scenario noted in the Introduction were discovered only in dynamic systems that were not ideal according to Sommerfeld–Kononeko or in systems with limited excitation. The fundamental feature of systems with limited excitation is the limited power of the energy source that excites the movement of a particular system. It is assumed that the power of the excitation source is comparable to the power consumed by the system itself. On the contrary, in so-called ideal systems no restrictions are imposed on the power of the excitation source. The fundamentals of the theory of systems with limited excitation were formulated in the monograph [13]. Various aspects of the dynamic behavior of various systems with limited excitation were studied in [14–18].

The main goal of this work is to demonstrate the possibility of implementing a generalized intermittency scenario in classical ideal dynamic systems (systems with unlimited excitation). The Lorenz system plays a unique role among ideal systems in chaotic dynamics. In this system in 1963, a new type of steady-state mode of a non-linear dynamic system, namely deterministic chaos, was firstly discovered.

The mathematical model of the Lorenz system can be written as a system of three differential equations [5; 19]:

$$\begin{aligned} \dot{x} &= \sigma(y - x); \\ \dot{y} &= rx - y - xz; \\ \dot{z} &= -bz + xy, \end{aligned} \tag{1}$$

here x, y, z are phase variables, and σ, r, b are some parameters. E.N. Lorenz derived a system of equations (1) at studying convection in a liquid layer, which is heated from below. In his work [19], the variable x characterizes the speed of rotation of the convection shafts, and the variables y, z are responsible for the horizontal and vertical temperature distribution, respectively. The parameter b is determined by the geometry of the convection cell, σ is the Prandtl number, which is the ratio of kinematic viscosity to the thermal diffusivity coefficient, and r is the Rayleigh number, dimensionless quantity that determines the behavior of the liquid under the influence of temperature.

Subsequently, it turned out that the Lorenz system of equations is applicable not only to the problem of convection in a layer of liquid, but also to describe the dynamics of many other physical systems. These include the convection problem in a closed loop, dissipative oscillator with inertial excitation, etc. [5; 20].

METHODS OF NUMERICAL CALCULATIONS AND MAIN NUMERICAL RESULTS

The system of differential equations (1) is nonlinear, therefore the main methods for studying the dynamic behavior of such a system are various numerical methods. Such methods are: Runge–Kutta method (both with constant and variable steps of numerical integration) for constructing phase portraits, Hénon method for constructing Poincaré sections and phase-parametric characteristics, the Benettin–

Galgani method for calculating Lyapunov characteristic exponents, computer algorithms for encoding images with shades of different brightness for constructing distributions of the natural invariant measure [5; 21; 22]. The general methodology for conducting such studies is described in [23].

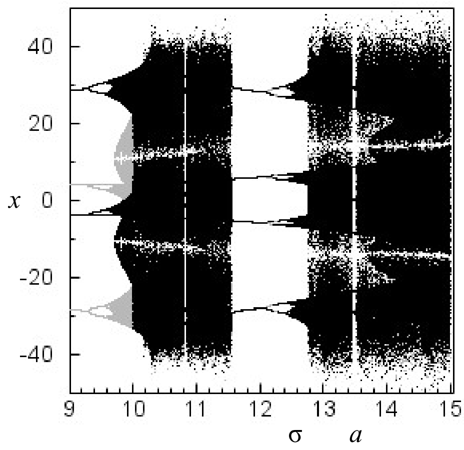
A characteristic feature of the Lorenz system is the existence of pairs of symmetric attractors, which could be either regular or chaotic. When the bifurcation parameter changes, a pair of separately existing symmetric attractors can "merge" into one attractor. Then, with a further change of the bifurcation parameter, a pair of attractors may arise again. Let us illustrate this process for some specific values of parameters of Lorenz system. Let the parameters of system (1) be respectively equal

$$r = 203.1, \quad b = \frac{8}{3}.$$

We take parameter b to be the same as in [19], and choose the Rayleigh number sufficiently large. We choose the Prandtl number σ as the bifurcation parameter.

Numerical calculations have shown that for values of the bifraction parameter $9 \le \sigma \le 10$, two symmetric attractors simultaneously exist in system (1). When $\sigma > 10$ these attractors "merge" and only one attractor remains in the system. In Fig. 1, *a* are shown the phase-parametric characteristics (bifurcation trees) of system (1). The values of the bifurcation parameter σ are plotted along the abscissa axis, and the values of the phase variable x are plotted along the ordinate axis. These characteristics were constructed using the Hénon method, when the plane $z = 170$ was chosen as secant plane.

In Fig. 1, *a*, at $9 \le \sigma \le 10$, the phase-parametric characteristic of one of the attractors is plotted in gray, and the other is plotted in black. The individual branches of bifurcation trees correspond to limit cycles of system (1). The points at which new individual branches of bifurcation trees appear are clearly visible. These points correspond to period doubling bifurcations of limit cycles. We note that such of period doubling bifurcations occur for both symmetric limit cycles for the same value of the bifurcation parameter σ. And, at last, the densely gray and the densely black areas of the bifurcation trees correspond to the chaotic attractors of the Lorenz system.



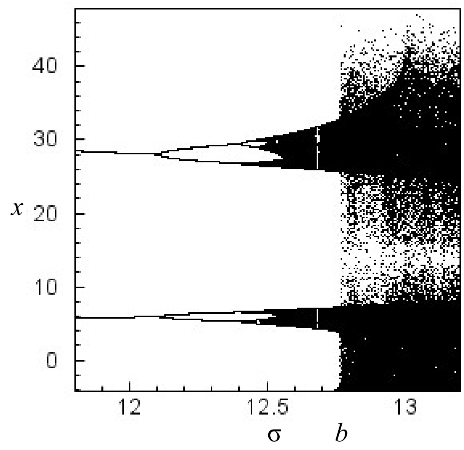


Fig. 1. Phase-parametric characteristics

In Fig. 2, *a* are constracted the phase portraits of two chaotic attractors of the Lorenz system at $\sigma = 10$. One of the attractors is plotted in gray, the other in black. Both chaotic attractors exist simultaneously, and each of them has its own basin of attraction. The transition from limit cycles to chaos for both attractors occurs according to Feigenbaum's scenario [1; 2]. These chaotic attractors are symmetric in the phase variables x and y. Both attractors have the same spectrum of Lyapunov characteristic exponents, and the maximum characteristic exponent must be positive. This is a necessary condition for the existence of chaos.

When $\sigma > 10$, the existing two chaotic attractors merge into one. As can be seen from the consideration of the bifurcation tree, for all values of $\sigma \in (10,15)$, there is a single attractor in the system, which can be either a limit cycle or a chaotic attractor. The phase portrait of such a "united" chaotic attractor, constructed at $\sigma = 10.01$, is shown in Fig. 2, *b*. Note that the spectrum of Lyapunov characteristic exponents of this attractor is practically no different from the corresponding spectrum of attractors are shown in Fig. 2, *a*.

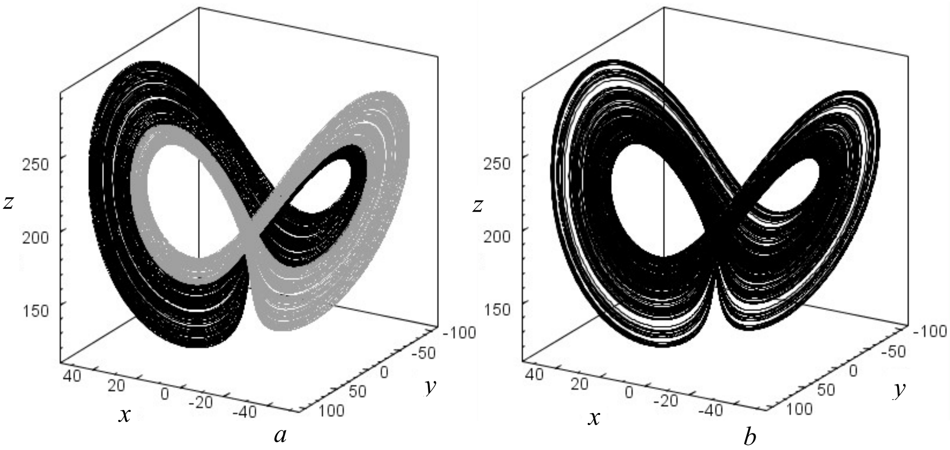


Fig. 2. Phase portrait of chaotic attractors at $\sigma = 10\,(a)$ and $\sigma = 10.01\,(b)$

Since the classical scenarios for the transition from regular attractors to chaotic attractors for the Lorenz system are quite well studied [4; 5; 21; 22], we will focus on the implementation of a new scenario of generalized intermittency.

For this, consider an enlarged fragment of the phase-parametric characteristic shown in Fig. 1, *b*. Let us study this phase-parametric characteristic in neighborhood of the bifurcation point $\sigma \approx 12.764$. Both in left and right semi-neighborhood of this bifurcation point in system (1) there are exist chaotic attractors (densely black areas of the bifurcation tree). However, as can be seen from Fig. 1, *b*, area of the densely black region of localization of the points of the bifurcation tree to the right of the point $\sigma \approx 12.764$ noticeably exceeds the corresponding area to the left of the bifurcation point. This feature of the phase-parametric characteristic is one of the signs of the implementation of the generalized intermittency scenario [6–12].

In Fig. 3, *a* is constracted the distribution of natural invariant measure over the phase portrait of the attractor at value $\sigma = 12.76$. The maximum Lyapunov characteristic exponent of this attractor, calculated using the method of Benettin et al., is positive and equal $\lambda_1 = 1.289$. The positivity of the maximum Lyapunov exponent is another evidence of chaotic nature of this attractor.

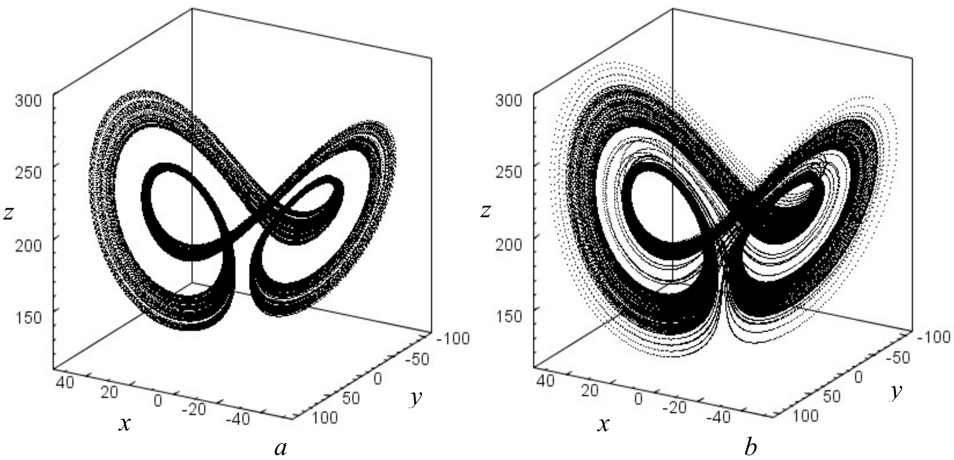


Fig. 3. Distribution of the natural invariant measure at $\sigma = 12.76$ (*a*) and $\sigma = 12.775$ (*b*)

At the value of bifurcation parameter σ increases, one hard bifurcation happens at point $\sigma \approx 12.764$. After which the chaotic attractor that existed at $\sigma < 12.764$ disappears and a chaotic attractor of a different type appears in the system (1). The transition from a chaotic attractor of one type to a chaotic attractor of another type occurs according to the scenario of generalized intermittency. In Fig. 3, *b* is constructed distribution of the natural invariant measure over phase portrait of the new chaotic attractor at $\sigma = 12.775$. The attractor that arises after passing the bifurcation point has a significantly larger positive maximum Lyapunov characteristic exponent. This characteristic exponent is equal $\lambda_1 = 1.71$. Such a noticeable increase in the value of the maximum characteristic exponent after passing the bifurcation point is another sign of the implementation of the scenario of generalized intermittency. The movement of the trajectory along the phase portrait of the arising chaotic attractor consists of two phases, so called, rough laminar and turbulent. Rough laminar phase corresponds to chaotic wanderings of the trajectory in the region of localization of the disappeared chaotic attractor (densely black area in Fig. 3, *b*). For turbulent phase corresponds to departures of the trajectory to more distant regions of the phase space (gray points on the distribution of the invariant measure in Fig. 3, *b*). Such transitions from the rough laminar phase of motion to the turbulent phase occur an infinite number of times. Note that the moment of transition from the rough laminar phase to the turbulent phase and the moment of return of the trajectory from the turbulent phase to the rough laminar phase are unpredictable. Also note, the time during which the trajectory is in a rough laminar phase significantly exceeds duration of time, in which trajectory in turbulent phase.

The scenario of generalized intermittency at transitions “a chaotic attractor of one type a chaotic attractor of another type” can also be identified by studying Poincaré sections. In Fig. 4, *a*, *b*, using the Hénon method, Poincaré sections of chaotic attractors at $\sigma = 12.76$ and at $\sigma = 12.775$ are constructed. Here $z = 200$ plane is selected as the secant plane. Both Poincaré sections have a quasi-ribbon structure and represent chaotic sets of individual points. Note that the quasi-ribbon structure is typical for chaotic attractors of the Lorenz system. As can be seen from Fig. 4, *b*, the structure of the Poincaré section of this attractor includes all fragments of the Poincaré section of the chaotic attractor shown in Fig. 4, *a*.

These fragments of the Poincaré section of the attractor at $\sigma = 12.76$ form a rough laminar phase of the attractor at $\sigma = 12.775$. Accordingly, in the structure of the Poincaré section of the attractor at $\sigma = 12.775$, new points additionally appear, which form the turbulent phase.

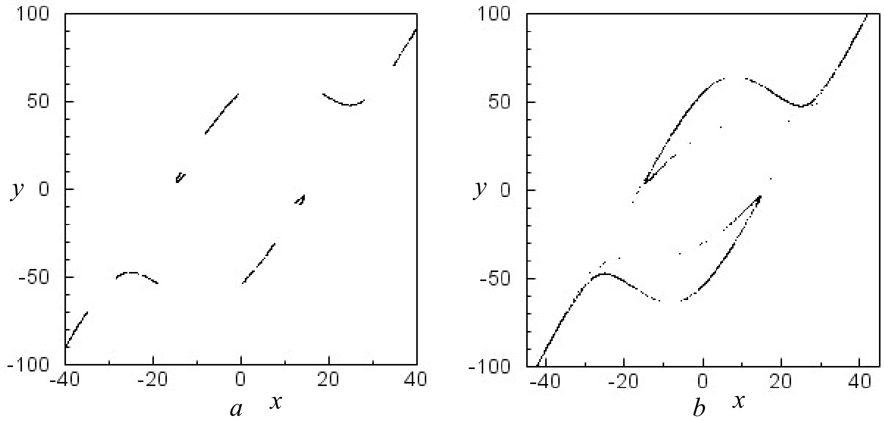


Fig. 4. Poincaré section at $\sigma = 12.76$ (*a*) and $\sigma = 12.775$ (*b*)

The implementation of the generalized intermittency scenario can be found when studying bifurcations in other parameters, in particular for r (the Rayleigh number in the classical work of Lorenz [19]). Let the parameters of system (1) be respectively equal

$$\sigma = 10, \; b = \frac{8}{3}.$$

And now we choose the Rayleigh number r as the bifurcation parameter. The technique for performing numerical calculations is similar to that used when studying bifurcations with respect to the parameter σ.

In Fig. 5, *a* are constructed parts of two phase-parametric characteristics of

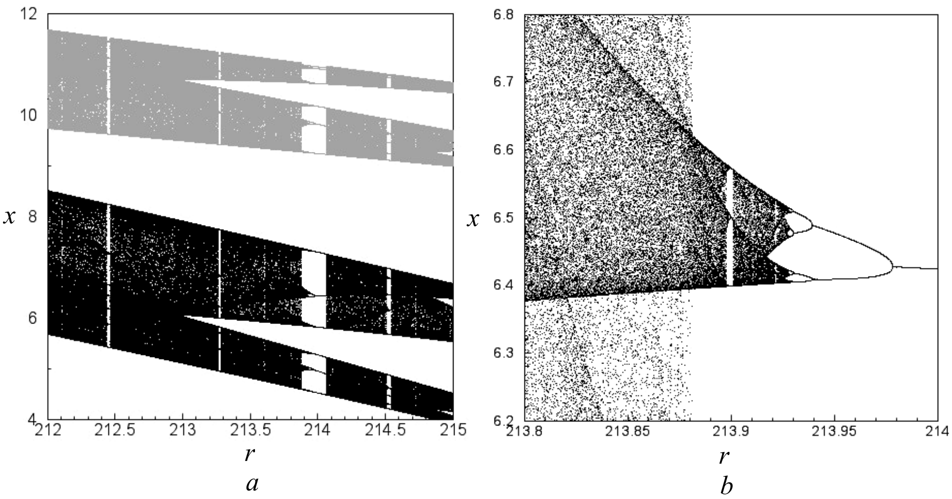


Fig. 5. Phase-parametric characteristics

system (1). These phase-parametric characteristics, “gray” and “black,” correspond to two attractors existing in the system. These two attractors exist in the system simultaneously. Moreover, the only possible situation is when either two limit cycles or two chaotic attractors simultaneously exist in the system. A careful study of Fig. 5, *a* allows us to identify a number of transitions to chaos according to Feigenbaum's scenario, which occur as the parameter r decreases. In turn, as the parameter r increases, transitions to chaos occur according to the Manneville–Pomeau scenario. We especially emphasize that all bifurcations of these two attractors occur at the same values of the parameter r.

However, we are primarily interested in the implementation of the generalized intermittency scenario. To do this, we will construct a significantly enlarged fragment of the "black" phase-parametric characteristic. Such a fragment is shown in Fig. 5, *b*. At decreasing the values of parameter r in Fig. 5, *b* is clear visible the process of branching of individual branches of the bifurcation tree. As is known, such a process describes the transition to chaos through a cascade of bifurcations of period doubling of the limit cycles. After the appearance of a chaotic attractor, with a further decrease in the values of the parameter r, a sufficiently small window of periodicity arises in which the cascade of bifurcations of period doubling again repeats and a chaotic attractor arises again.

After the appearance of a chaotic attractor, with a further decrease in the values of the parameter r, a sufficiently small window of periodicity appears in which the period-doubling bifurcation sequence again repeats and a chaotic attractor appears again. Finally, when $r \approx 213.885$, one of the signs of the implementation of the generalized intermittency scenario can be seen on the phase-parametric characteristic. Namely, a significant increase in the region of chaos in the phase-parametric characteristic. We studied bifurcations only for one of existing chaotic attractors of the system. However, the same processes can be seen when studying an enlarged fragment of the "gray" phase-parametric characteristic, that correspond to symmetric chaotic attractor.

In Fig. 6, *a* the distributions of the natural invariant measure are plotted over the phase portraits of a pair, symmetric with respect to *x* and *y*, attractors at $r = 213.89$. These symmetrical attractors are shown in black and gray. In Fig. 6, *b* shows the distributions of the natural invariant measure over the phase portraits of another pair of symmetric attractors at $r = 213.88$. In this figure, each of the symmetrical attractors is also shown in black and gray. Both pairs of constructed attractors are chaotic attractors, since they have positive maximum Lyapunov characteristic exponents. As shown by numerical calculations carried out using the method of Benettin et al., each of the chaotic attractors shown in Fig. 6, *a* have the same maximum exponent $\lambda_1 = 0.18$. Accordingly, the second pair of chaotic attractors has maximum exponent $\lambda_1 = 0.28$. So, when passing the bifurcation point $r \approx 213.885$, the second sign of the implementation of the generalized intermittency scenario is observed, namely a significant increase of the maximum Lyapunov characteristic exponent.

For a more visual description of the phases of generalized intermittency in Fig. 6, *c*, *d*, respectively, show significantly enlarged fragments of the distributions of invariant measures for one of the attractors before and immediately after the bifurcation point. Note that for the second of the symmetric attractors we will have a similar description of the phases of generalized intermittency. The rough laminar phase in Fig. 6, *d* form thicker black areas, the shape of which is similar to the areas of distribution of invariant measures in Fig. 6, *c*. The turbulent phase is formed by individual points between the "turns" of the rough laminar phase. As in the case of studying bifurcations with respect to σ parameter, a picture typical for generalized intermittency can also be found when studying the similarities of Poincaré sections before and after the bifurcation point $r \approx 213.885$.

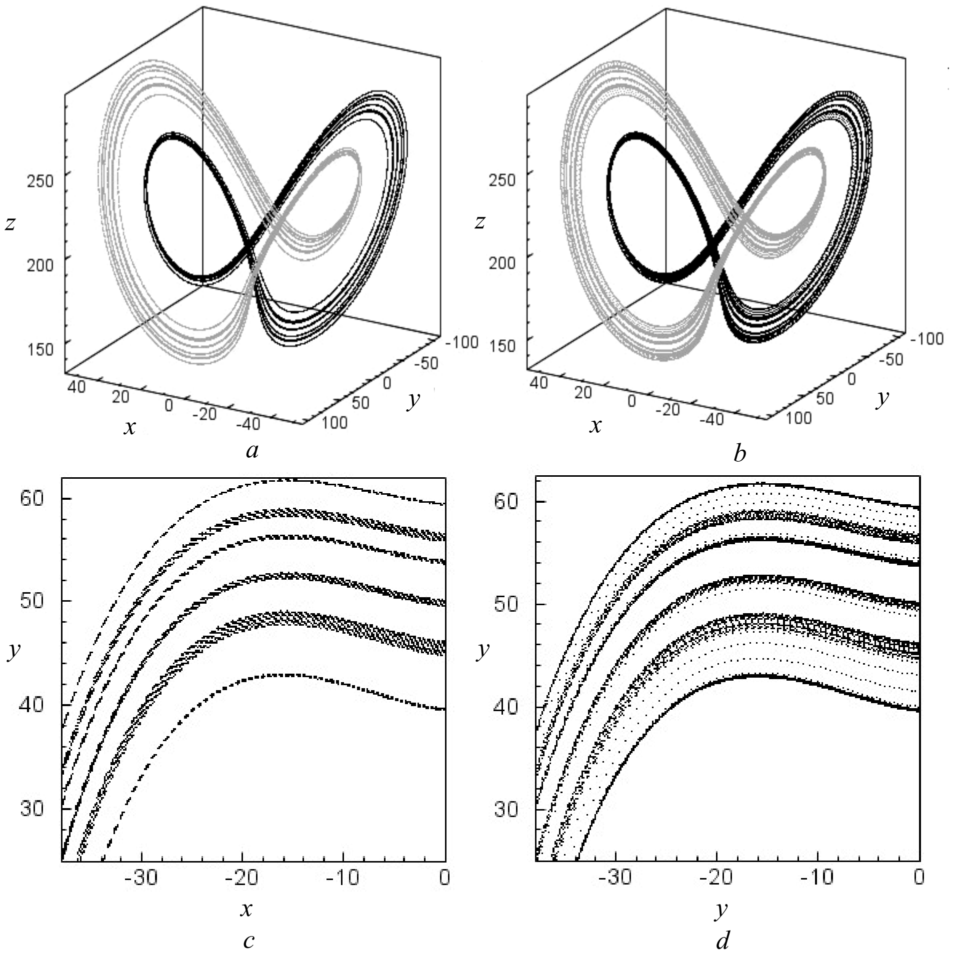


Fig. 6. Distribution of the natural invariant measure at $\sigma = 213.89(a)$ and $\sigma = 213.88(b)$; fragments of projections at $\sigma = 213.89(c)$ and at $\sigma = 213.88(d)$

CONCLUSIONS

Thus, for the classical ideal dynamic Lorenz system, the implementation of transitions from "a chaotic attractor of one type to a chaotic attractor of another type" was established according to the scenario of generalized intermittency. It is shown that such a scenario can be realized both in the case of existence of a single attractor in the Lorenz system, and in the case of existence of a pair of symmetric attractors.

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INFORMATION ON THE ARTICLE

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УЗАГАЛЬНЕНІ СЦЕНАРІЇ ПЕРЕХОДУ ДО ХАОСУ В ІДЕАЛЬНИХ ДИНАМІЧНИХ СИСТЕМАХ / О.О. Горчаков, О.Ю. Швець

Анотація. Виявлено реалізацію нового сценарію переходу до хаосу в класичній системі Лоренца. Описано ознаки наявності реалізації сценарію узагальненої переміжності для динамічних систем. Побудовано та детально проаналізовано фазово-параметричні характеристики, показники Ляпунова, розподіли інваріантних мір та перерізи Пуанкаре, які підтверджують реалізацію сценарію узагальненої переміжності в ідеальній системі Лоренца.

Ключові слова: ідеальна динамічна система, регулярний і хаотичний атрактори, сценарій узагальненої переміжності.