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IDENTIFICATION OF NONLINEAR SYSTEMS WITH PERIODIC EXTERNAL ACTIONS (Part I)

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Abstract. The problem of identifying nonlinear systems with periodic external actions is considered in the article. The number of such actions in the system is not limited, and these actions can be either additive or multiplicative. We use a time series of observed system variables to calculate unknown equation coefficients. The proven theorem allows us to separate the unknown coefficients of the system into variables and constants. The proposed computational procedure allows us to avoid possible errors caused by the discrete nature of observable time series. Identification of zero coefficients is carried out in two ways, eliminating erroneous zeroing of the terms of the equations. The method is illustrated with a numerical example of identifying a chaotic system with periodic external actions.

Keywords: identification, ordinary differential equation, external action, periodic coefficient, constant coefficient.

INTRODUCTION

Nonlinear systems with external actions occur in the study of many real objects and processes. Such systems are widespread, for example, in biology [1–3], ecology [4], epidemiology [5], mechanical engineering [6–8], and electrical engineering [9].

A significant amount of research is devoted to constructing models of non-autonomous systems, including those that involve periodic external actions. In [4], for example, non-autonomous models of the “predator-prey” type are studied for almost periodic systems that are used in bioinformatics, social networks, and wireless sensor networks. Study [6] is devoted to the analysis of the influence of external periodic force on the behavior of the model of single-degree-of-freedom vibro-impact system. In this work the conditions for the transition of the model from the chaotic to the regular regime have been studied. In paper [7], a nonlinear non-autonomous dynamic model of a quarter vehicle with nonlinear spring and damping was studied. The influences of the damping coefficient, external action amplitude and frequency on the dynamic responses were analyzed. It was established that the system could have chaotic, quasi-periodic or periodic motion. A study of a bio-reactor model with periodic nutrient forcing is presented in [10]. The paper [11] investigates the global behaviors of the logistic system with periodic impulsive perturbations. The authors formulate condition under which the system may have periodic solution. A wide class of non-autonomous models is described and analyzed in [12]. An increasing interest in non-autonomous systems was admitted in [13]. This survey introduces basic concepts and tools for non-autonomous dynamical systems and their application to various biological models. Investigation [14] is devoted to the analysis of various modes of the oscillator with periodic external action.

Additionally, extensive research has been conducted to address one of the specific instances of the inverse problem [15], which is the identification of non-autonomous systems based on the observed variables. For example, neural networks were used in [16; 17] to solve this problem. The study [18] proposes a method for constructing a nonlinear, non-autonomous model with a hyperbolic linear part. The articles [19; 20] consider various approaches to identifying systems of differential equations with an additive external action. A similar problem for systems of difference equations was solved in [21].

FORMULATION OF THE PROBLEM

In the works mentioned above, the problem of identifying a system with a known model structure and additive external action is typically addressed. We are attempting to solve a more general problem. Namely, we propose a method for finding external actions, both additive and multiplicative, without limiting their quantity. This task is complicated by the lack of information regarding which coefficients of the differential equation are constant and which are periodic (representing external actions). Also, the task can become more difficult when we study systems with deterministic chaos. As is known [22], the behavior of such systems essentially depends on the initial conditions.

Consider a system consisting of ordinary differential equations (ODEs) of the form

$$\dot{x}_i = \sum_{j=0}^{m} c_{ij}(t) f_j(\mathbf{x}), \tag{1}$$

where $i = 1,...,n;$ $\mathbf{x} = \{x_1(t),...,x_n(t)\}$. We assume that the right-hand sides of equation (1) satisfy the conditions for the existence and uniqueness of the solution on a certain time interval when $t \in [0;t_e],$ $t_e > 0$.

The coefficients in equation (1) can be of three types: $c_{ij}(t) = \text{const}$, $c_{ij}(t) = 0$, and $c_{ij}(t) = p_{ij}(t),$ where $p_{ij}(t)$ is a continuous periodic function with a period T. This function represents an external action on the system. Each equation of the system can have many periodic coefficients that correspond to external actions. Moreover, all periodic coefficients in each equation have the same period. Generally speaking, the external actions of equation (1) are multiplicative. If, for example, $f_0(\mathbf{x}) = 1$, then the external action $c_{i0}(t)$ becomes additive.

A method is proposed for solving the following problem. We assume that the functions $x_i(t)$, $\dot{x}_i(t)$, and $f_{ij}(\mathbf{x})$ in equation (1) are known. It is necessary to define the following:

1. Determine which coefficients of equation (1) are periodic, which are constant, and which are zero. We assume that equation (1) has at least one coefficient of each type. At the same time, the number of coefficients of each type is limited only by the total number of coefficients in the equation, which is $m+1$.

2. Find the period of the functions $p_{ij}(t)$.

3. Find the values of the constant coefficients.
4. Find the form of the functions $p_{ij}(t)$.

METHOD

To solve this problem, we will use the following approach. Consider the case when, in equation (1), $c_{ij}(t) = \text{const} \quad \forall j \in 0,...,m$. Then, to find these coefficients, we can use a system of $m+1$ linear algebraic equations (SLAE) compiled for $m+1$ time points $t_0,...,t_m$:

$$\begin{cases} \dot{x}_i(t_0) = c_{i0} f_0(\mathbf{x}(t_0)) + c_{i1} f_1(\mathbf{x}(t_0)) + ... + c_{im} f_m(\mathbf{x}(t_0)), \\ \dot{x}_i(t_1) = c_{i0} f_0(\mathbf{x}(t_1)) + c_{i1} f_1(\mathbf{x}(t_1)) + ... + c_{im} f_m(\mathbf{x}(t_1)), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{x}_i(t_m) = c_{i0} f_0(\mathbf{x}(t_m)) + c_{i1} f_1(\mathbf{x}(t_m)) + ... + c_{im} f_m(\mathbf{x}(t_m)). \end{cases} \tag{2}$$

The SLAE (2) coefficients can be calculated using the well-known relation [23]:

$$\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}, \tag{3}$$

where

$$\mathbf{C} = \begin{bmatrix} c_{i0} \\ c_{i1} \\ ... \\ c_{im} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} f_0(\mathbf{x}(t_0)) \;\; f_1(\mathbf{x}(t_0)) \; ... f_m(\mathbf{x}(t_0)) \\ f_0(\mathbf{x}(t_1)) \;\; f_1(\mathbf{x}(t_1)) \; ... f_m(\mathbf{x}(t_1)) \\ ... \\ f_0(\mathbf{x}(t_m)) \;\; f_1(\mathbf{x}(t_m)) \; ... f_m(\mathbf{x}(t_m)) \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \dot{x}_i(t_0) \\ \dot{x}_i(t_1) \\ ... \\ \dot{x}_i(t_m) \end{bmatrix}.$$

To calculate constant coefficients using formula (3), it is sufficiently to do the following:

1. Select arbitrary moments of time $t_0,...,t_m$.
2. Form matrix $\mathbf{A}$ and vector $\mathbf{B}$ for these moments. We assume that matrix A is not singular.
3. Use formula (3) to obtain vector $\mathbf{C}$.

Note that, since in this case all the $c_{i0},...,c_{im}$ coefficients are constant, it is sufficiently to perform all the listed above operations for one set of $t_0,...,t_m$.

If at least one of the required coefficients is a function of time $c_{ij}(t) = \text{var}$, it is necessary to find its values at each point in the time interval $t \in [0; t_e]$. Using the procedure described above for some arbitrary set of $t_0,...,t_m$, we will obtain some value of coefficient $c_{ij}(t)$. At the same time, it is not known to which point in time from the interval $[0; t_e]$ this value corresponds. Thus, the function $c_{ij}(t)$ cannot be constructed. As will be shown below, to eliminate this uncertainty when calculating periodic coefficients using formula (3), it is sufficiently to impose some conditions on the moments of time for which matrix $\mathbf{A}$ and vector $\mathbf{B}$ are formed.

Let the time moments for which the SLAE (2) is formed obey the relations:

$$t_1 = t_0 + \tau, \;\; t_2 = t_0 + 2\tau, \; ... \; , \;\; t_m = t_0 + m\tau; \;\; t_0 \geq 0, \;\; \tau > 0, \;\; t_m \leq t_e. \tag{4}$$

Theorem. Let equation (1) has constant coefficients and periodic coefficients with a period of T. Additionally, let the SLAE be formed in the form of equation (2) for the time moments, subject to the conditions (4) and when $\tau = T$. When solving the SLAE using relation (3), we obtain a vector $\mathbf{C}$, which consists of the values of the periodic coefficients of equation (1) at time t_0 as well as the values of the constant coefficients of this equation.

Proof. Let us first assume that in equation (1), one of the coefficients, for example, $c_{i0}(t)$, is periodic with a period T, while the remaining coefficients are constant. Then the SLAE (2) will take the following form:

$$\begin{cases} \dot{x}_i(t_0) = c_{i0}(t_0) f_0(\mathbf{x}(t_0)) + c_{i1} f_1(\mathbf{x}(t_0)) + \ldots + c_{im} f_m(\mathbf{x}(t_0)), \\ \dot{x}_i(t_1) = c_{i0}(t_1) f_0(\mathbf{x}(t_1)) + c_{i1} f_1(\mathbf{x}(t_1)) + \ldots + c_{im} f_m(\mathbf{x}(t_1)), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{x}_i(t_m) = c_{i0}(t_m) f_0(\mathbf{x}(t_m)) + c_{i1} f_1(\mathbf{x}(t_m)) + \ldots + c_{im} f_m(\mathbf{x}(t_m)). \end{cases} \quad (5)$$

Let us form a matrix $\mathbf{A}$ and vector $\mathbf{B}$ to solve system (5). To do this, we choose the moments of time according to the condition stated in the theorem, which is condition (4). If at the same time $\tau = T$, then, since the function is periodic with a period of T, we get

$$c_{i0}(t_0) = c_{i0}(t_1) = \ldots = c_{i0}(t_m), \quad (6)$$

where

$$t_1 = t_0 + T,\ t_2 = t_0 + 2T, \ldots,\ t_m = t_0 + mT.$$

That is, in the system (5), the periodic coefficient $c_{i0}(t)$ becomes constant. Therefore, applying formula (3) to solve SLAE (5) is correct. The resulting vector of coefficients of the system (5) will include the values of the constant coefficients of equation (1) and the value of the periodic coefficient at time moments (6).

According to the conditions of the problem, all periodic coefficients of equation (1) have the same period. Therefore, the above reasoning is valid if equation (1) has more than one periodic coefficient. That is, when forming the SLAE of the form (2) and considering relations (4) at $\tau = T$, all periodic coefficients of equation (1) will have constant values that correspond to the time moment t_0. Then, by solving the SLAE (2), we can determine these constant values of the periodic coefficients of equation (1) as well as the values of the constant coefficients of equation (1). This completes the proof.

Corollary. To separate the desired coefficients into constant and periodic ones, it is sufficient to form two systems of the form (2) for two values of t_0: t_{01} and t_{02} while considering conditions (4). If $\tau = T$, $t_{01} \neq t_{02}$, and $|t_{01} - t_{02}| \neq kT$ $(k = 1,2,\ldots)$, then when solving these two SLAEs, we obtain the same values for the constant coefficients in equation (1).

This Corollary was used to construct the identification algorithm described in the next section. It is also necessary to note an important special case that may arise with an arbitrary choice of t_{01} and t_{02} values in the proposed algorithm and which may lead to incorrectness of results. This situation will be considered in the Special case section.

NUMERICAL RESULTS

On the basis of proven Theorem and its Corollary, an algorithm was developed that can be divided into two stages. At the beginning, we can address the first three points of the formulated problem. In the second stage, the form of the functions $p_{ij}(t)$ is determined.

To illustrate the method, we used system (7), which was built based on the well-known Rössler system [24]:

$$\begin{cases} \dot{x}_1 = -x_2 - x_3, \\ \dot{x}_2 = x_1 - dx_2, \\ \dot{x}_3 = c_{30}(t) + c_{33}(t)x_3 + c_{36}(t)x_1x_3, \end{cases} \qquad (7)$$

where

$$d = 0.15,\ c_{30}(t) = 0.5 + 0.4\sin\left(\frac{2\pi t}{T}\right),\ c_{33}(t) = -20,\ c_{36}(t) = 5 + 2.5\sin\left(\frac{2\pi t}{T} - \frac{\pi}{2}\right).$$

As we can see, the third equation of the system has two external actions: an additive one $c_{30}(t)$ and a multiplicative one $c_{36}(t)$. The period of external actions was taken to be $T = 2.11\,s$. The system was solved over an interval of 100 s with a step size of $\Delta t = 0.01s$. Fig. 1 shows the time series of the variables in the system (7), and Fig. 2 displays its phase trajectories.

The object of study is the third equation of the system (7). We will identify it based on the general structure of the form (8), which includes a second-degree polynomial on the right-hand side:

$$\dot{x}_3 = c_{30}(t) + c_{31}(t)x_1 + c_{32}(t)x_2 + c_{33}(t)x_3 + c_{34}(t)x_1^2 + c_{35}(t)x_1x_2 + \\ + c_{36}(t)x_1x_3 + c_{37}(t)x_2^2 + c_{38}(t)x_2x_3 + c_{39}(t)x_3^2. \qquad (8)$$

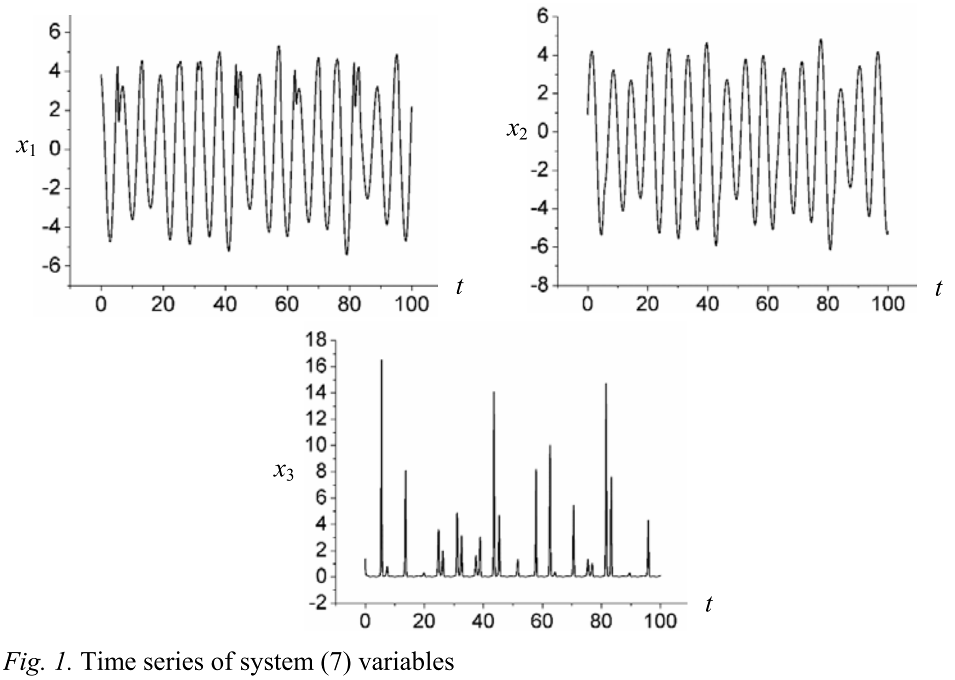

Fig. 1. Time series of system (7) variables

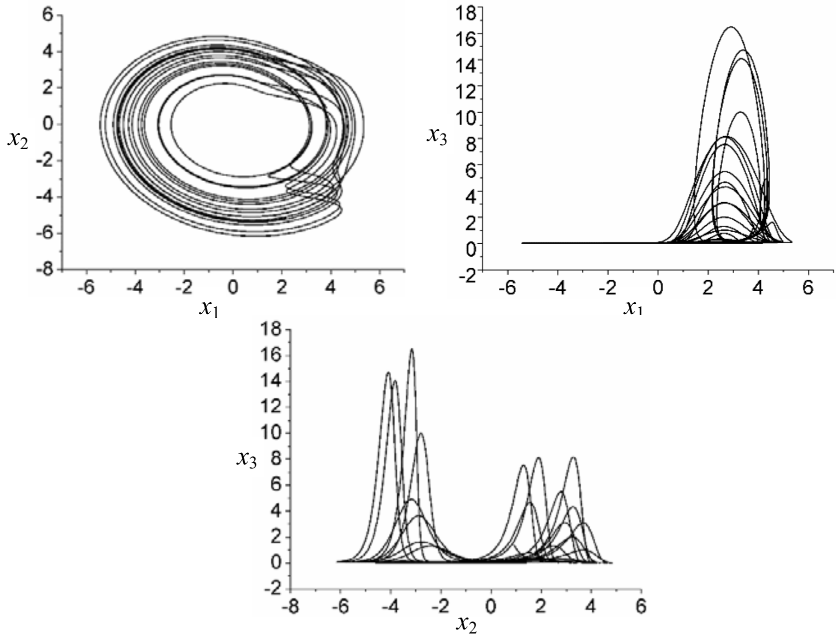

Fig. 2. Phase trajectories of system (7)

The solution to the formulated problem can be significantly simplified if it is possible to first estimate the period of external actions on the system. This possibility exists, for example, with resonance [25]. In this case, the period of external action can be estimated based on the period of oscillation of the observed variables. It is easy to show that for a chaotic system (7) such an approach will not lead to proper results. Considering that the external actions of this system are included in its third equation, then, first of all, these actions can affect the shape of the function $x_3(t)$. As a result, this function can become periodic. But, as follows from Fig. 1, this function has no periodicity. Also, the period of external actions cannot be estimated based on an analysis of the shape of the functions $x_1(t)$ and $x_2(t)$. For example, Fig. 3 shows the time series $x_2(t)$ and $v_{36}(t)$, where the latter is the variable component of the external action $c_{36}(t)$, $v_{36} = 2.5\sin(2\pi t/T - \pi/2)$. It is obvious that the quasi-period of a function $x_2(t)$ does not coincide with the period of $v_{36}(t)$ and is not a multiple. Therefore, we cannot preliminarily estimate the period of external actions and thus simplify the solution of the problem. Moreover, due to the lack of information about the existence of external periodic action, we may erroneously assume that the model has only constant coefficients. Such an initial assumption may lead to the construction of an inadequate model.

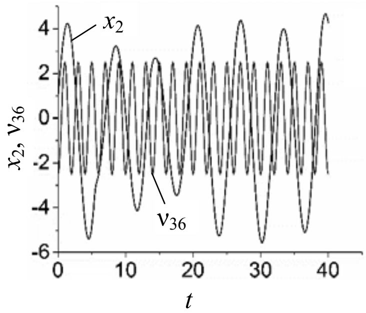

Fig. 3. Comparison of oscillation periods of functions $v_{36}(t)$ and $x_2(t)$

In general, the main steps of the first stage of the proposed algorithm in relation to equation (8) are as follows:

1. We set t_{01}, form the SLAE (2), and solve it by setting the values of τ within a certain range of $(\tau_b;\tau_e)$ which presumably includes the desired value of T. Thus, we obtain the values of all coefficients $c_{3j}(t)$.

2. Repeat step 1 for $t_{02} \neq t_{01}$.

3. For each coefficient $c_{3j}(t)$, we find a value τ at which

$$\delta_j = \sqrt{(c^1_{3j} - c^2_{3j})^2} \to \min , \quad (9)$$

where c^1_{3j} and c^2_{3j} represent the values of the coefficient $c_{3j}(t)$ obtained for t_{01} and t_{02}, respectively.

The calculations were carried out for $t_{01} = 0.15s$, $t_{02} = 0.4s$, $\tau_b = 1.0s$, $\tau_e = 11.5s$. The calculation results are shown in Table 1. The first line of Table 1 shows the τ values for which relation (9) is satisfied. Since the calculations involve discrete time series instead of continuous functions $x_i(t)$, there is a possibility of errors when calculating (9). Therefore, in order to obtain more complete information for analyzing the results, lines 2–5 of Table 1 show the τ values for which the δ_j value is closest to zero. As the line number increases, the δ_j value also increases.

Table 1. The first line shows the τ values at which δ_j takes on the least values

№	**The τ values calculated for the coefficients of equation (8) at $\delta_j \to \min$**									
	$c_{30}(t)$	$c_{31}(t)$	$c_{32}(t)$	$c_{33}(t)$	$c_{34}(t)$	$c_{35}(t)$	$c_{36}(t)$	$c_{37}(t)$	$c_{38}(t)$	$c_{39}(t)$
1	2.08	**8.44**	**2.11**	**8.44**	**8.44**	**2.11**	8.78	**2.11**	**8.44**	**2.11**
2	10.10	**2.11**	**8.44**	10.05	**2.11**	**8.44**	4.16	**8.44**	**2.11**	**8.44**
3	5.26	**4.22**	**10.55**	**10.55**	**4.22**	**10.55**	6.67	**4.22**	**10.55**	1.31
4	10.03	**10.55**	**4.22**	**2.11**	**10.55**	2.05	4.06	**10.55**	**6.33**	**4.22**
5	3.34	9.38	4.71	**4.22**	8.10	**4.22**	5.44	8.37	**4.22**	5.42

Based on the data presented in Table 1, the following conclusions can be drawn.

1. In the columns corresponding to the coefficients $c_{31}(t)$, $c_{32}(t)$, $c_{33}(t)$, $c_{34}(t)$, $c_{35}(t)$, $c_{37}(t)$, $c_{38}(t)$, $c_{39}(t)$, the predominant values are $\tau = 2.11s$ or multiples: $4.22s$, $6.33s$, $8.44s$ and $10.55s$ (these values are highlighted in bold in the table). Based on the corollary of the theorem, it can be argued that these coefficients are constant. Such regularity is not observed for the coefficients $c_{30}(t)$ and $c_{36}(t)$, suggesting that these coefficients are variable.

2. In addition, we can infer from the table that the search value for the period of external action is $T = 2.11s$, and consequently, $4.22s = 2T$, $6.33s = 3T$, $8.44s = 4T$, $10.55s = 5T$.

The values of the coefficients c^1_{3j} and c^2_{3j}, which satisfy relation (9) and were used to fill the first line of Table 1, are indicated in Table 2. An examination of these values suggests that equation (8) has only one non-zero constant coefficient, c_{33}. The final conclusion can be reached after further analysis.

Table 2. Calculated values of the constant coefficients of equation (8)

t_0	**Coefficients**							
	c_{31}	c_{32}	c_{33}	c_{34}	c_{35}	c_{37}	c_{38}	c_{39}
$t_{01} = 0.15s$	3.417*10^{-5}	5.070*10^{-5}	-20.019	2.399*10^{-6}	6.222*10^{-6}	3.682*10^{-7}	-9.514*10^{-4}	9.320*10^{-2}
$t_{02} = 0.4s$	-2.017*10^{-5}	-5.214*10^{-7}	-20.022	-1.450*10^{-6}	2.168 *10^{-7}	2.094*10^{-7}	3.027*10^{-4}	4.599*10^{2}

Moreover, it will be possible to answer the fourth point of the formulated problem, which is to determine the form of the functions $c_{30}(t)$ and $c_{36}(t)$. The second part of the algorithm is dedicated to solving this problem. With the already known value of T, the SLAE (2) is formed for t_0 varying within a certain range. By solving the SLAE at all points within this range, we can evaluate the values of all coefficients (both constant and variable) at these specific time intervals. The time series of certain coefficients, obtained from the calculation for $10\ s \le t_0 \le 30\ s$, are shown in Fig. 4. In the graphs, the calculated values of the coefficients are indicated by $c_{30}^c, c_{31}^c, \ldots$ The graphs in this figure have singularities, i.e., some points where the values of c_{ij} differ significantly from neighboring points. This occurs because the matrix $\mathbf{A}$ formed for calculating the coefficients of c_{ij}, in this case, has a determinant $\det \mathbf{A} \to 0$.

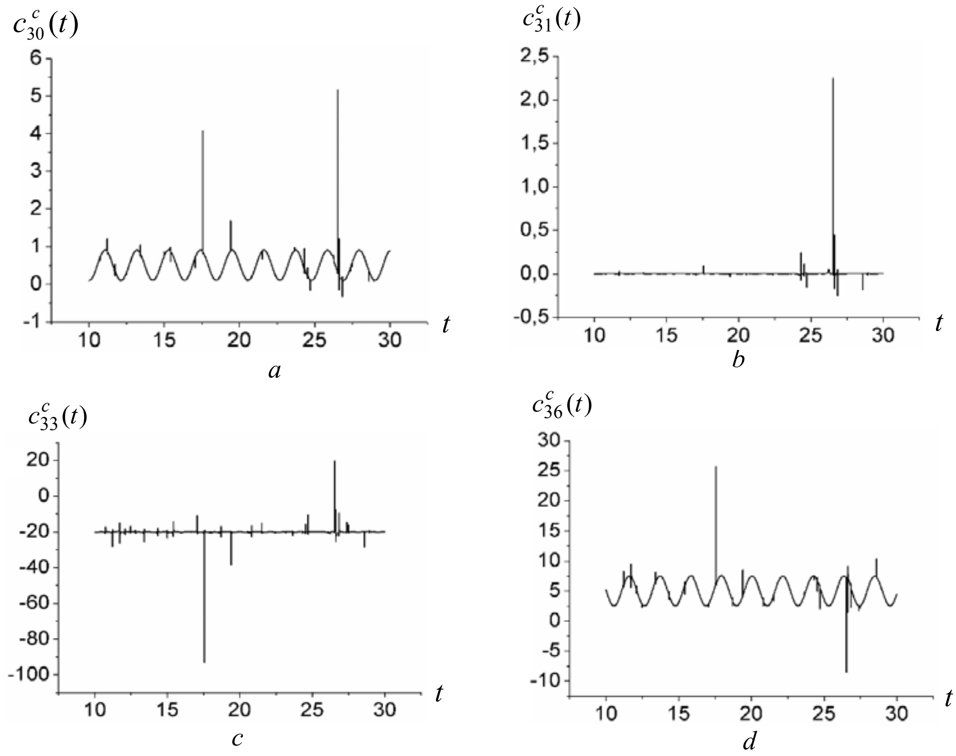

Fig. 4. Time series of $c_{30}^c(t)$, $c_{31}^c(t)$, $c_{33}^c(t)$, $c_{36}^c(t)$ obtained as a result of the calculation

We also note that, for example, in Fig. 4, *b*, $c_{31}^c \approx 0$ at points where the singularity does not occur. Here, graphs for some of the coefficients from Table 2 are not shown since all of them, except for $c_{33}(t)$, have a similar form to Fig. 4, *b*. That is, the values of these coefficients are close to zero. This fact confirms the preliminary assessment based on the data in Table 2, namely: $c_{31} = c_{32} = c_{34} = c_{35} = c_{37} = c_{38} = c_{39} = 0$.

Re-identification of equation (8) using non-zero coefficients $c_{30}(t)$, $c_{33}(t)$, $c_{36}(t)$ allowed us to obtain the time series, as shown in Fig. 5, *a*, *b*, *c*. Fig. 5, *d* shows the original and identified time series of the $c_{30}(t)$ coefficient on the interval $21.2s \le t \le 22.2s$. This figure illustrates the proximity of these series, except for points with a singularity.

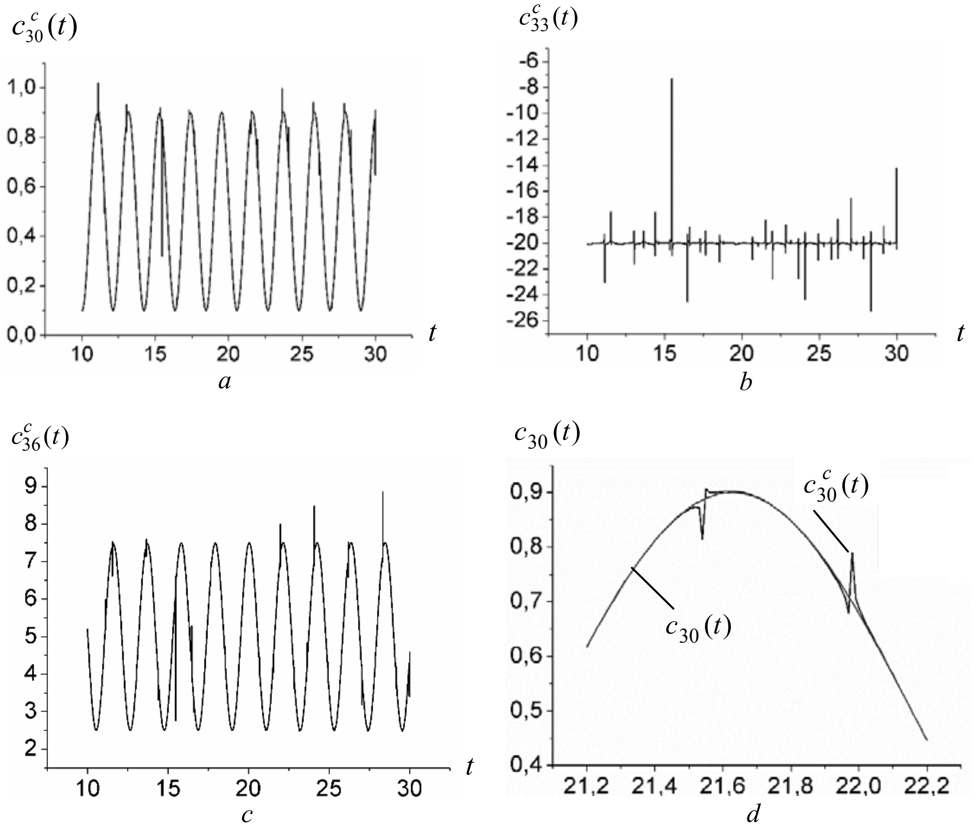

Fig. 5. Time series of calculated coefficients: a — $c_{30}^{c}(t)$; b — $c_{33}^{c}(t)$; c — $c_{36}^{c}(t)$; d — initial $c_{30}(t)$ and calculated $c_{30}^{c}(t)$ time series on the interval $21.2s \le t \le 22.2s$

Fig. 6 shows the time series of errors: $\Delta c_{30} = c_{30}^{c} - c_{30}$, $\Delta c_{33} = c_{33}^{c} - c_{33}$, $\Delta c_{36} = c_{36}^{c} - c_{36}$, which allows us to visually estimate the accuracy of identification.

SPECIAL CASE

As noted in the Method section, for a certain set of calculation parameters the result may be incorrect. This situation is possible when a periodic function $c_{ij}(t)$ has the period T and the following conditions are met

$$c_{ij}(t_{01}) = c_{ij}(t_{02}), \ |t_{01} - t_{02}| = \frac{T}{a}, \ (a = 2,3,...). \quad (10)$$

Let us illustrate the features of the algorithm application in this case with an example. Let the system (7) have the following parameters:

$$d = 0.15, \ c_{30}(t) = 1 + \sin\left(\frac{2\pi t}{T}\right), \ c_{33}(t) = -20, \ c_{36}(t) = 5, \ T = 2s. \quad (11)$$

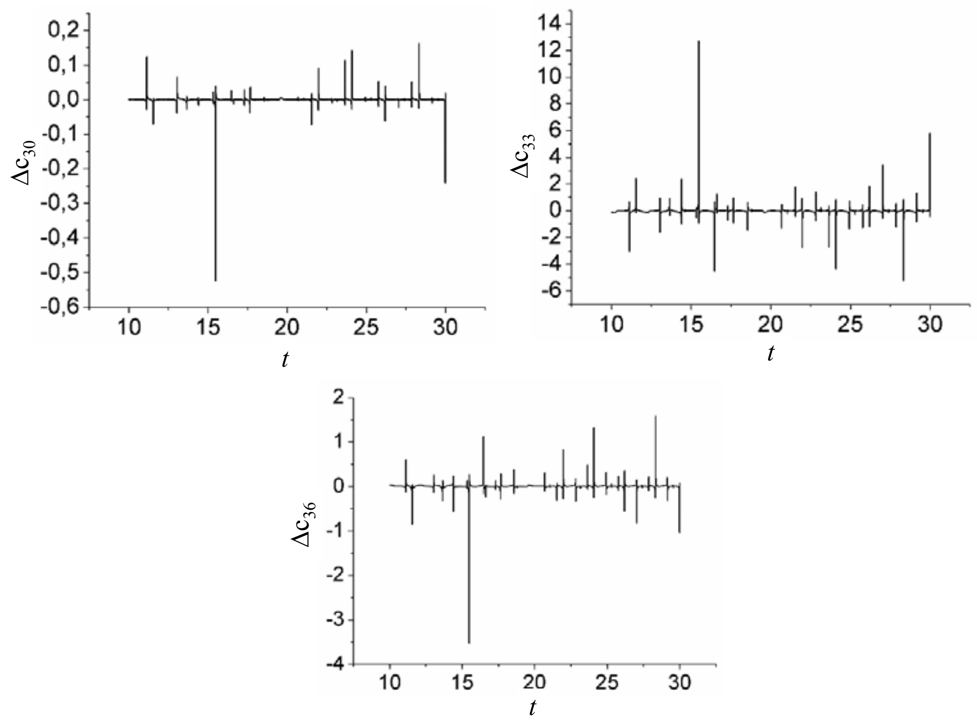

Fig. 6. Time series of errors

The system was solved over an interval of 100 s with a step size of $\Delta t = 0.01s$. Identification was carried out according to the algorithm described above for $t_{01} = 1s$, $t_{02} = 2s$. The input action graph is shown in Fig. 7. As follows from (11) and illustrated by the graph, the following relationships hold:

$$c_{30}(t_{01}) = c_{30}(t_{02}), \quad |t_{01} - t_{02}| = \frac{T}{2} = 1s .$$

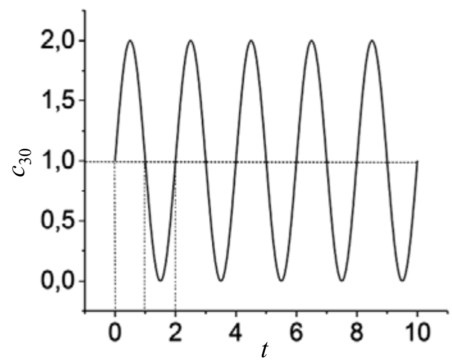

Fig. 7. Time series of external action in (11)

It can be noted that these relationships correspond to conditions (10). As a result of applying the algorithm, the data presented in Table 3 were obtained.

An analysis of these data similar to the analysis of Table 1 may lead us to incorrect conclusions.

1. Since all columns of Table 3 contain values $\tau = 1, 2, \ldots$, all the coefficients are constant.

2. We can also mistakenly assume that $T = 1s$, and the values $2s, 3s, \ldots, 10s$ are multiples.

Obviously, both of these conclusions are incorrect. But this result can be easily corrected by changing the calculation parameters of the algorithm to violate the conditions (10). In this case, the value $t_{02} = 2.01s$ was used instead of $t_{02} = 2s$. That is, the moment of time t_{02} was shifted by 1 step compared to the previous case. The calculation results are given in Table 4.

Table 3. The same as in Table 1 for system (7) with parameters (11)

№	The τ values calculated for the coefficients of equation (8) at $\delta_j \to \min$									
	$c_{30}(t)$	$c_{31}(t)$	$c_{32}(t)$	$c_{33}(t)$	$c_{34}(t)$	$c_{35}(t)$	$c_{36}(t)$	$c_{37}(t)$	$c_{38}(t)$	$c_{39}(t)$
1	**1.00**	**9.00**	**4.00**	**7.00**	**3.00**	**10.00**	6.07	**3.00**	**4.00**	**6.00**
2	**7.00**	**2.00**	**9.00**	**1.00**	**2.00**	**4.00**	**6.00**	**8.00**	**2.00**	5.72
3	**9.00**	**10.00**	**8.00**	**6.00**	**10.00**	**8.00**	**2.00**	**4.00**	**9.00**	5.64
4	**2.00**	**3.00**	**10.00**	**9.00**	**1.00**	**5.00**	**5.00**	**2.00**	**8.00**	5.99
5	**3.00**	**1.00**	**2.00**	**5.00**	**5.00**	**9.00**	**9.00**	**7.00**	3.42	**10.00**

Table 4. The same as in Table 3 with $t_{02} = 2.01s$

№	The τ values calculated for the coefficients of equation (8) at $\delta_j \to \min$									
	$c_{30}(t)$	$c_{31}(t)$	$c_{32}(t)$	$c_{33}(t)$	$c_{34}(t)$	$c_{35}(t)$	$c_{36}(t)$	$c_{37}(t)$	$c_{38}(t)$	$c_{39}(t)$
1	9.07	**10.00**	**4.00**	**6.00**	**2.00**	**10.00**	**2.00**	**8.00**	**4.00**	**6.00**
2	2.49	**2.00**	**10.00**	**2.00**	**10.00**	**4.00**	**6.00**	**4.00**	7.65	6.10
3	9.47	4.51	**8.00**	8.40	7.50	**8.00**	**8.00**	**2.00**	**2.00**	8.53
4	5.72	4.84	**2.00**	8.97	**8.00**	**2.00**	**4.00**	**10.00**	**8.00**	5.67
5	0.58	**8.00**	**6.00**	**8.00**	**4.00**	2.28	**10.00**	**6.00**	**10.00**	**10.00**

The data in Table 4 already allow us to draw the correct conclusions.

1. In the identified equation, the coefficient c_{30} is periodic, the other coefficients are constant.

2. The function $c_{30}(t)$ has a period $T = 2s$ and the values $4s, 6s, 8s, 10s$ are multiples.

It should be noted that in a real study, repeating the identification with changed t_{02} is not mandatory to obtain the correct result. It is sufficient to apply the last stage of the algorithm based on the data in Table 3, namely, to try to obtain the form of the input action with already known possible values of T. For this purpose, the SLAE (2) is formed taking into account the conditions (6) for t_0 varying within a certain range. By solving the SLAE at all points within this range, we can evaluate the values of c_{ij} coefficients for all these points. Thus, we obtain the function $c_{ij}(t)$.

On the contrary, when conditions (10) are met, the condition $c_{ij}(t_0) = c_{ij}(t_1) = ... = c_{ij}(t_m)$ is met only for specific t_0. If we choose $T = 1s$, then, for example, at $t_0 = 0s$ we will get according to (6) $t_1 = 1s$. As it is seen from Fig. 7, $c_{30}(0s) = c_{30}(1s)$. However, at $t_0 = 0.5s$ we will get $t_1 = 1.5s$ and $c_{30}(0.5s) \neq c_{30}(1.5s)$. Thus, we cannot construct a function $c_{ij}(t)$ for all t_0 within a given time interval.

This feature becomes apparent when we perform the second stage of the algorithm. Time series of all coefficients at $T = 1s$ were calculated. Two of them are shown in Fig. 8, *a* and 8, *b*. For comparison, Fig. 8, *c* and 8, *d* show the graphs obtained for the same functions, but at $T = 2s$. The choice of the correct T value when considering these four graphs is obvious.

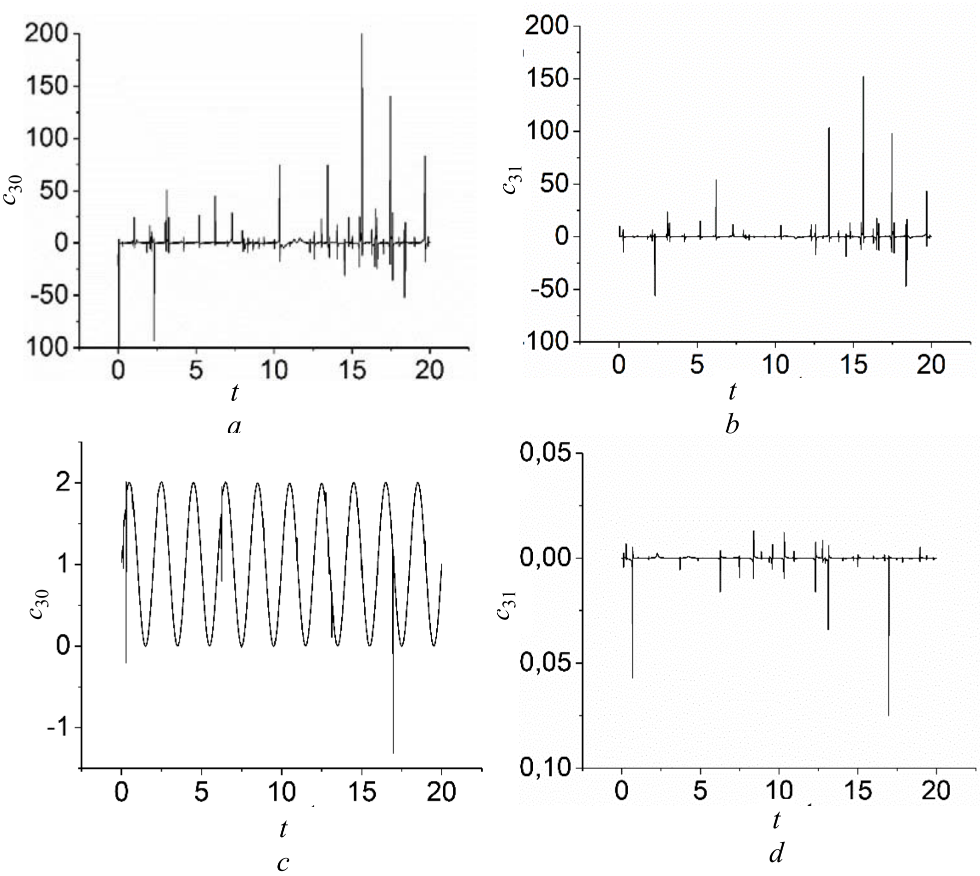

Fig. 8. Time series of calculated coefficients $c_{30}(t)$ and $c_{31}(t)$: *a* and *b* for $T = 1\ s$; *c* and *d* for $T = 2\ s$

CONCLUSIONS

The proven theorem and its corollary make it possible to solve the inverse problem with many unknowns. Such unknowns can be the number and values of constant ODE coefficients, the number, period and forms of external actions. The latter can be both additive and multiplicative. The number of external actions in each equation of the system is unlimited. The only restriction is that all the external actions in each equation must have the same period. The method allows us to detect unknown periodic actions that cannot be identified based on the form of observed variables.

To solve the formulated problem, it is not necessary to know in advance which coefficients in the ODE system's equations are variables, constants, or zeros. To use proposed method for solving formulated problem it is sufficiently to have time series of observed variables.

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ІДЕНТИФІКАЦІЯ НЕЛІНІЙНИХ СИСТЕМ З ПЕРІОДИЧНИМИ ЗОВНІШНІМИ ДІЯМИ (Частина І) / В.Г. Городецький

Анотація. Розглянуто проблему ідентифікації нелінійних систем з періодичними зовнішніми діями. Кількість таких дій у системі не обмежена, і ці дії можуть бути як адитивними, так і мультиплікативними. Для обчислення невідомих коефіцієнтів рівнянь використано часові ряди спостережуваних змінних системи. Доведена теорема дозволяє розділити невідомі коефіцієнти системи на змінні та сталі. Запропонована обчислювальна процедура дозволяє уникнути можливих помилок, спричинених дискретністю спостережуваних часових рядів. Ідентифікацію нульових коефіцієнтів виконано двома способами, що виключає помилкове обнулення членів рівнянь. Метод ілюстровано числовим прикладом ідентифікації хаотичної системи з періодичними зовнішніми діями.

Ключові слова: ідентифікація, звичайне диференціальне рівняння, зовнішня дія, періодичний коефіцієнт, сталий коефіцієнт.