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IDENTIFICATION OF NONLINEAR SYSTEMS WITH PERIODIC EXTERNAL ACTIONS (Part II)

V. GORODETSKYI

Abstract. The article presents the results of the study, which is a continuation of the author's previous research. This paper considers more complex problems in identifying nonlinear systems with periodic external actions. The article shows that the previously proposed method is applicable when the periods of external actions in the same differential equation may differ. At the same time, the ratio between the values of the periods can be both integer and fractional. The conditions under which this is possible are formulated. These conditions are based on the theorem proved in the previous work. Part of this study is devoted to the problem of identification of a chaotic system with an external non-sinusoidal action. To create such an external action, a function with three harmonic components was used. A numerical experiment confirmed the effectiveness of the algorithm in this case as well.

Keywords: identification, ordinary differential equation, external action, periodic coefficient, constant coefficient.

INTRODUCTION

As is known, non-autonomous mathematical models are widely used to describe various physical processes [1; 2]. The construction of such models can be reduced to the so-called inverse problem [3]. In this case, the model is built on the basis of information about the output of the system, that is, the problem of system identification is solved. In this case, the usual formulation of the problem assumes the presence in the system equations of additive periodic actions and information about the structure of the system [4–6]. The task becomes more complex when the structure of the system is unknown and external actions can be either additive or multiplicative. At the same time, the number of such actions may be not limited. The solution of the mentioned problem was proposed in [7]. This study is a development of the author's previous work and demonstrates additional capabilities of the method introduced in [7].

NOTATIONS AND SOME PREVIOUS RESULTS

Using the notations from [7], we consider a system of ordinary differential equations (ODE) of the form

$$\dot{x}_i = \sum_{j=0}^{m} c_{ij}(t) f_j(\mathbf{x}), \tag{1}$$

$$i = 1,...,n; \quad \mathbf{x} = \{x_1(t),...,x_n(t)\}. \quad t \in [0; t_e], \quad t_e > 0.$$

In equation (1), we consider the time functions $x_i(t)$ to be known and the coefficients $c_{ij}(t)$ to be unknown. In this case, any of the coefficients can be either constant or a continuous time function of a period T.

If in equation (1) all the coefficients $c_{ij}(t) = const,$ then to find them we can apply the well-known relation:

$$\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}, \tag{2}$$

where $\mathbf{C}$ is the vector of the required coefficients of equation (1), and $\mathbf{B}$ is the vector of values $\dot{x}_i(t_k),\ k = 0,...,m,$ $\mathbf{A}$ is the matrix of function $f_j(\mathbf{x}(t_k))$ values, $j = 0,...,m$. In [7], a theorem was proven according to which relation (2) can be used to calculate the coefficients of equation (1) if the moments of time t_j are subject to the relations:

$$t_1 = t_0 + \tau,\ t_2 = t_0 + 2\tau,\ ...,\ t_m = t_0 + m\tau;\ t_0 \geq 0,\ \tau > 0,\ t_m \leq t_e \tag{3}$$

and wherein $\tau = T$. Based on this theorem, the algorithm described in [7] was constructed. That is, if conditions (3) are met and $\tau = T$, then applying formula (2) for any t_0, we will obtain the exact values of all constant coefficients of the identified equation. Therefore, if $\tau = T$, then for two different t_0: t_{01} and t_{02} the relation

$$\delta_j = |c_{ij}^1 - c_{ij}^2| \to \min, \tag{4}$$

must be satisfied, where c_{ij}^1 and c_{ij}^2 are the values of the coefficient $c_{ij}(t)$ obtained for t_{01} and t_{02}, respectively. In order to avoid errors possible for a specific value of t_0, intervals of t_{01} and t_{02} values are used in calculations. The application of the algorithm is illustrated in [7] using the example of identifying an equation with additive and multiplicative periodic actions having the same period.

GENERALIZATION OF THE PROPOSED METHOD

It is easy to show that the method can be effective for solving more complex problems. Let an equation of the form (1) have two external periodic actions with periods T_1 and T_2, respectively. Let also there exist q_1 and q_2, $q_1, q_2 \in 1,2,3,...$ and $T < t_e$, such that

$$q_1 T_1 = q_2 T_2 = T, \tag{5}$$

where T is the least common multiple of T_1 and T_2. That is, we have period T, which is common for both external actions. Therefore, the condition of the theorem from [7] is met.

Identification of equations with an integer ratio of periods of external actions

Let us consider the case when relation (4) is satisfied, and at the same time $T_1/T_2 \in 2,3,...$ or $T_2/T_1 \in 2,3,....$ As an example of using the method, consider identification of a system

$$\begin{cases} \dot{x}_1 = -x_2 - x_3, \\ \dot{x}_2 = x_1 - dx_2, \\ \dot{x}_3 = c_{30}(t) + c_{33}(t)x_3 + c_{36}(t)x_1x_3, \end{cases} \quad (6)$$

obtained on the basis of the well-known Rössler system [8]. The system has the following coefficients:

$$d = 0.15,\ c_{30}(t) = 0.5 + 0.4\sin\left(\frac{2\pi t}{T_0}\right),$$

$$c_{33}(t) = -20,\ c_{36}(t) = 5 + 2.5\sin\left(\frac{2\pi t}{T_6} - \frac{\pi}{2}\right),\ T_0 = 2s,\ T_6 = 4s.$$

It is obvious that for the external actions of the system (6), we have $T_6/T_0 = 2$. That is, condition (5) is satisfied with $T_6 = T$. Fig. 1 shows the time series of the variables of the system (6) and Fig. 2 shows its phase trajectories.

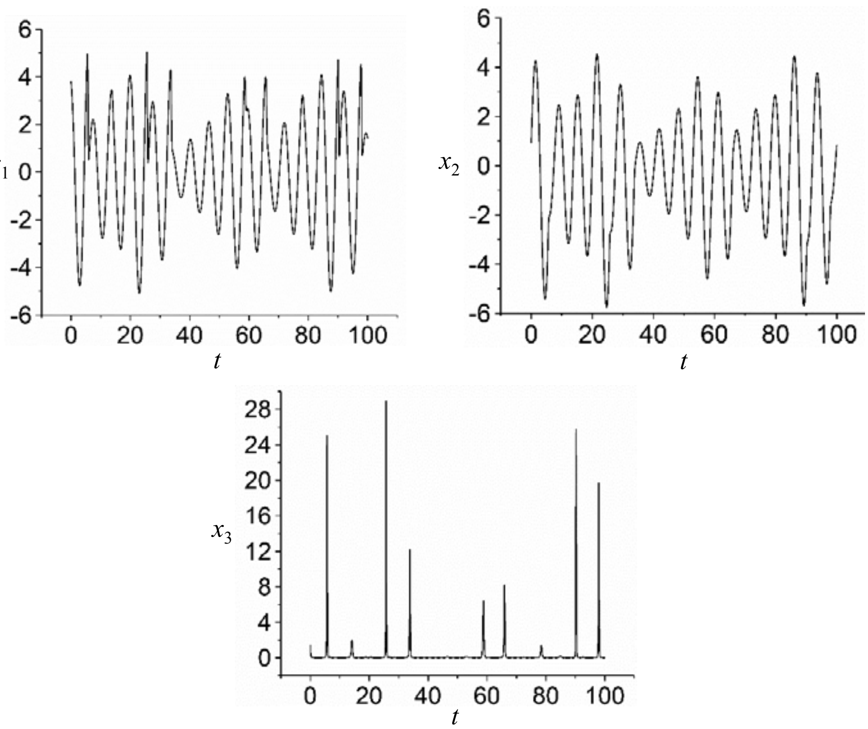

Fig. 1. Time series of system (6) variables

To identify the third equation of system (6), its general structure was chosen in the form of a polynomial of the second degree:

$$\dot{x}_3 = c_{30}(t) + c_{31}(t)x_1 + c_{32}(t)x_2 + c_{33}(t)x_3 + c_{34}(t)x_1^2 + c_{35}(t)x_1x_2 +$$

$$+ c_{36}(t)x_1x_3 + c_{37}(t)x_2^2 + c_{38}(t)x_2x_3 + c_{39}(t)x_3^2. \quad (7)$$

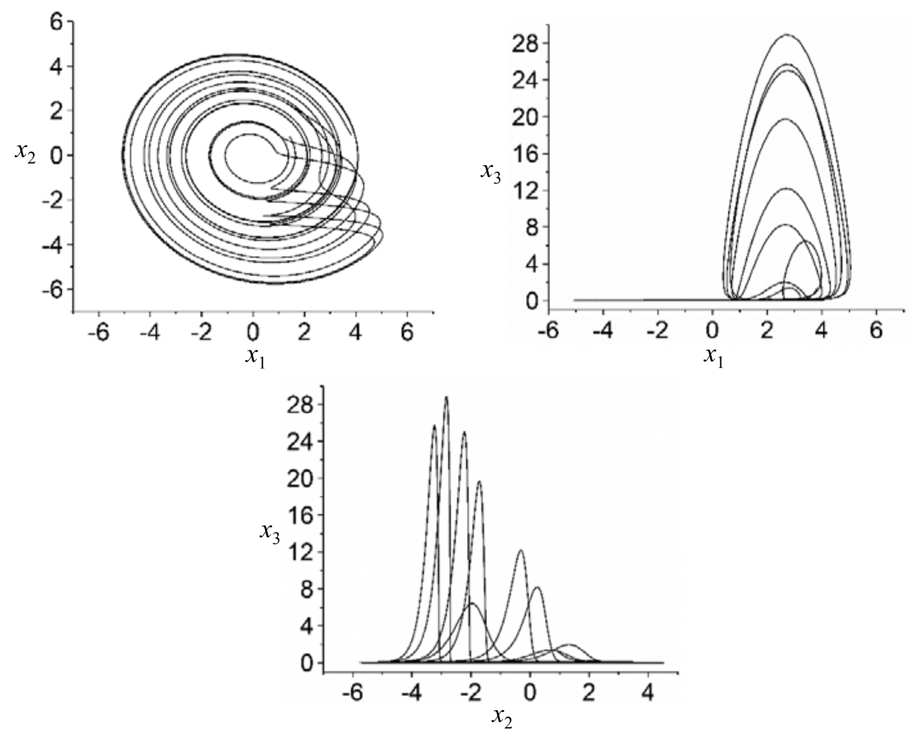

Fig. 2. Phase trajectories of system (6)

The results of applying the algorithm [7] are presented in Table 1.

Table 1. The τ values for which the δ_j value is closest to zero. The first row shows the τ values at which δ_j takes on the least values. As the row number increases, the δ_j value also increases

№	**The τ values calculated for the coefficients of equation (7) at $\delta \to \min$**									
	$c_{30}(t)$	$c_{31}(t)$	$c_{32}(t)$	$c_{33}(t)$	$c_{34}(t)$	$c_{35}(t)$	$c_{36}(t)$	$c_{37}(t)$	$c_{38}(t)$	$c_{39}(t)$
1	2.18	4.75	**8.00**	9.10	**4.00**	**8.00**	6.18	**4.00**	**8.00**	2.71
2	4.31	**4.00**	**4.00**	**8.00**	**8.00**	**4.00**	0.58	**8.00**	**4.00**	**8.00**
3	1.88	**8.00**	1.81	5.30	3.91	6.25	6.16	10.33	4.44	8.33
4	1.26	3.41	7.98	4.89	9.76	3.95	4.09	7.99	4.01	5.39
5	7.99	1.06	3.82	3.20	5.78	6.07	6.19	3.98	7.98	6.48

In Table 1, the τ values that are repeated or multiples are highlighted in bold. The theorem in article [7] suggests that they can correspond to the real value of the period. It also follows from the theorem that the presence of such values in a certain column of the table indicates that this coefficient is constant.

Note that when $\tau = T$, relation (4) must be satisfied. That is, the τ values highlighted in bold (which correspond to the real value of the period) must be located in the top row of the table. However, this is not observed for some coefficients, which is explained by the possible presence of computational errors [7].

As can be seen from Table 1, the least period obtained as a result of the calculation is $T = 4s$. The value of $8s$ from the table obviously corresponds to $2T$. Now that the period T of the external actions is known, it is possible to determine the form of all functions $c_{3j}(t)$ using the final part of the algorithm [7]. To do this, we form matrix $\mathbf{A}$ and vector $\mathbf{B}$ for system (2) taking into account relations (3) with $\tau = T$ and solve system (2) for t_0 values from a certain range.

Thus, we obtain the values of the functions $c_{3j}(t)$ at all points in this range. Fig. 3 shows the time series $c_{30}^c(t)$, $c_{31}^c(t)$, $c_{33}^c(t)$, $c_{36}^c(t)$, obtained as a result of the calculation at the interval $t_0 \in 0,...,20\,s$. The figure shows time series of coefficients of different types: variables $c_{30}^c(t)$ and $c_{36}^c(t)$, constant zero $c_{31}^c(t)$, and constant non-zero $c_{33}^c(t)$. The numerical values of the constant coefficients can be estimated from the form of the obtained time series $c_{3j}(t)$. More accurate values can be obtained using their values obtained by solving system (2) for t_{01} and t_{02}, for which $\tau = T$ or τ is a multiple of T, see Table 2 in [7].

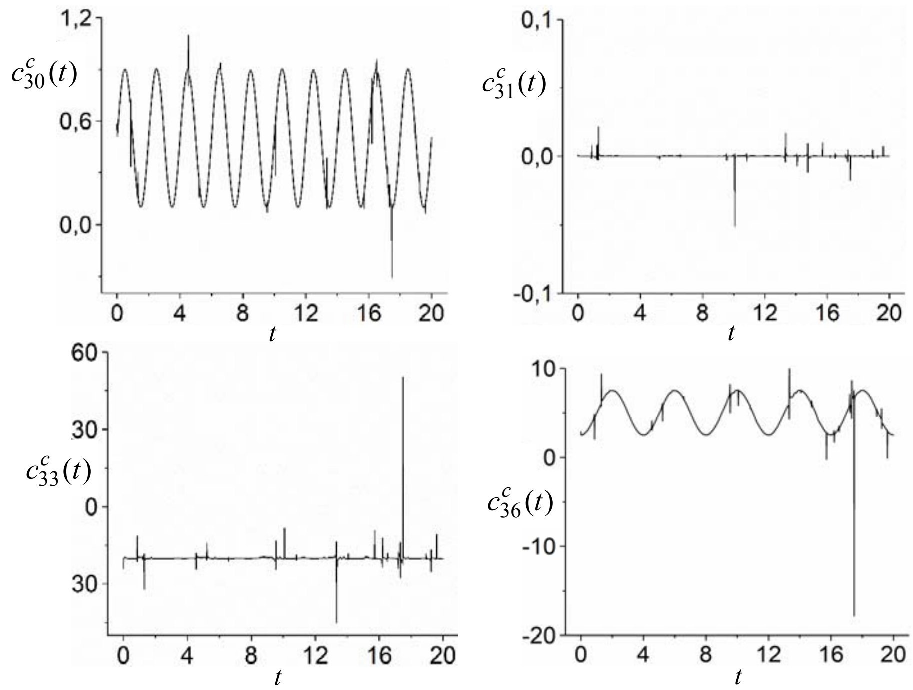

Fig. 3. Time series of calculated coefficients of the third equation of system (6). Calculation was performed using structure (7) of equation

Based on the graphs and estimated values of the constant coefficients, the zero coefficients can be eliminated. As a result, the structure of the general equation (7) is reduced to the structure of the third equation of system (6). Since we know T value and have a simplified equation structure, we can re-identify the equation. Time series of the obtained coefficients are presented in Fig. 4.

Note that according to the calculations (Table 1), condition (3) was satisfied for $\tau = 4\,s$. This value obviously corresponds to the period of external action T_6. At the same time, as expected, the value of T_0 was not determined as a result of the calculation. However, after comparing the form of the external action $c_{30}^c(t)$ with $c_{36}^c(t)$ in Fig. 3 or 4, it can be argued that $T_0 = T_6/2 = 2\ s$.

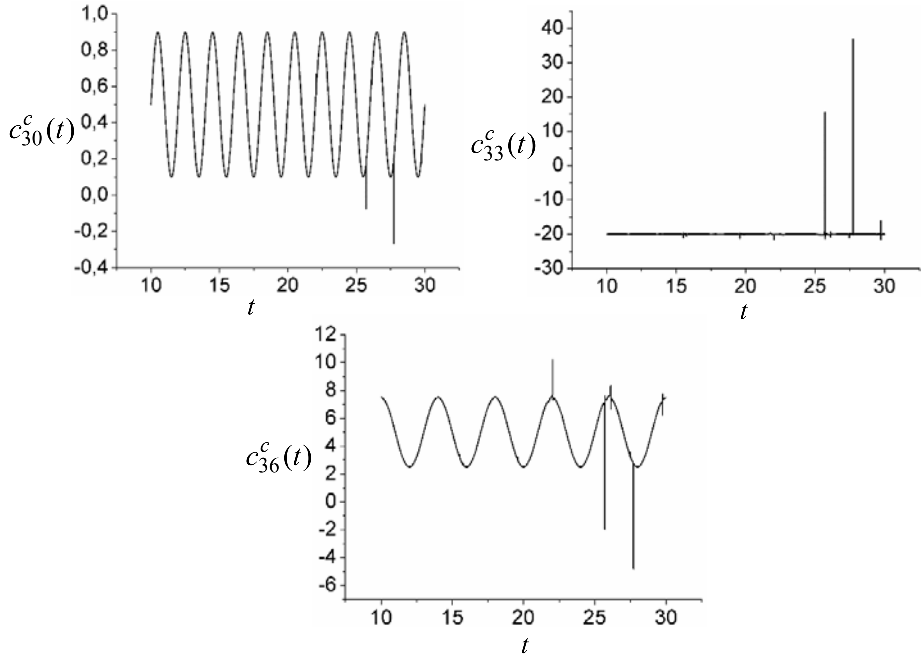

Fig. 4. Time series of calculated coefficients of the third equation of system (6). Calculation was performed using simplified structure of the equation

Identification of equations with a fractional ratio of periods of external actions

Let's consider a more general case when the periods of actions T_1 and T_2 in the equation are subject to condition (5) and in this case T_1/T_2, $T_2/T_1 \notin 2,3,...$. Let the external actions in the third equation of the system (6) have the form:

$$c_{30}(t) = 0.5 + 0.4\sin\left(\frac{2\pi t}{T_0}\right),\ c_{36}(t) = 5 + 2.5\sin\left(\frac{2\pi t}{T_6} - \frac{\pi}{4}\right),\ T_0 = 3s,\ T_6 = 2s. \quad (8)$$

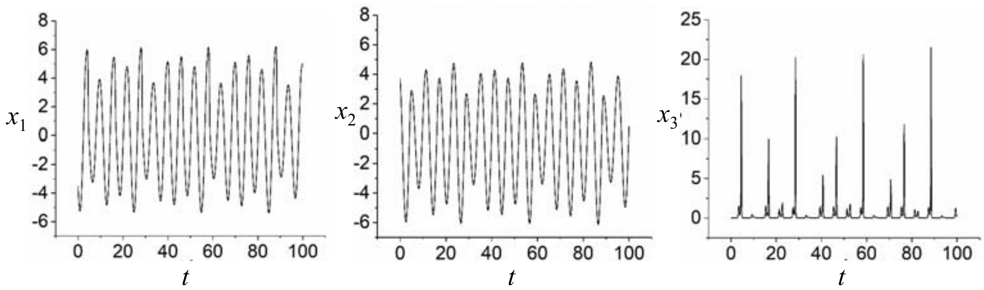

Fig. 5. Time series of variables of system (6) with external actions (8)

Time series and phase trajectories of system (6) under input actions (8) are presented in Fig. 5 and 6, respectively.

Obviously, with $q_1 = 2$ and $q_2 = 3$ we get $q_1 T_0 = q_2 T_6 = T$, $T = 6s$. That is, relation (5) is satisfied. Then, considering T to be the only period of external actions in the equation, we can apply the theorem and algorithm from [7]. System (6) with external actions (8) was solved on an interval of $100s$ with a step of $\Delta t = 0.01s$. According to the algorithm, the initial times $t_{01} = 0.15s$, $t_{02} = 0.4s$ were selected. The τ value was chosen from the interval $[\tau_b;\tau_e]$, $\tau_b = 1s$,

$\tau_e = 11s$. Table 2 presents the τ values for which relation (4) is satisfied. As can be seen from the table, in most of its columns the value $\tau = 6s$ is found (shown in bold in the table).

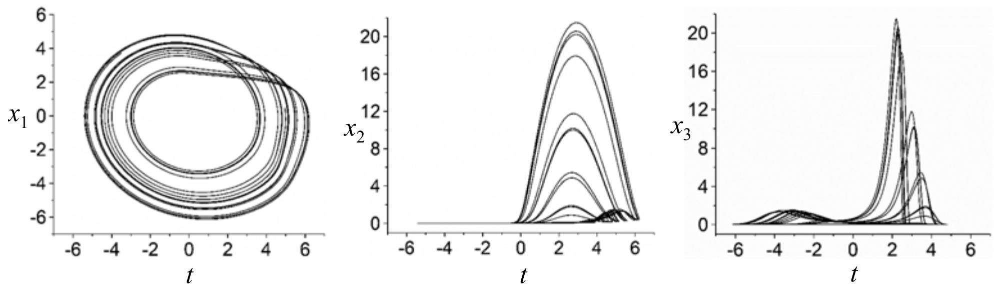

Fig. 6. Phase trajectories of system (6) with external actions (8)

Table 2. The same as in Table 1, for system (6) with external actions (8)

№	The τ values calculated for the coefficients of equation (7) at $\delta \to \min$									
	$c_{30}(t)$	$c_{31}(t)$	$c_{32}(t)$	$c_{33}(t)$	$c_{34}(t)$	$c_{35}(t)$	$c_{36}(t)$	$c_{37}(t)$	$c_{38}(t)$	$c_{39}(t)$
1	9.81	2.91	2.00	10.02	**6.00**	**6.00**	6.32	**6.00**	6.36	2.90
2	2.69	**6.00**	8.03	7.29	1.36	2.70	5.94	9.98	9.99	1.86
3	9.48	4.89	7.62	8.18	8.05	1.43	7.99	2.59	**6.00**	4.49
4	9.68	9.12	8.00	1.25	2.55	6.91	6.55	1.47	2.61	**6.00**
5	8.91	2.90	**6.00**	1.27	10.11	7.19	1.96	6.01	5.94	3.00

After an analysis similar to that carried out in the previous section and elimination of zero coefficients, we obtain time series of constant and variable coefficients presented in Fig. 7.

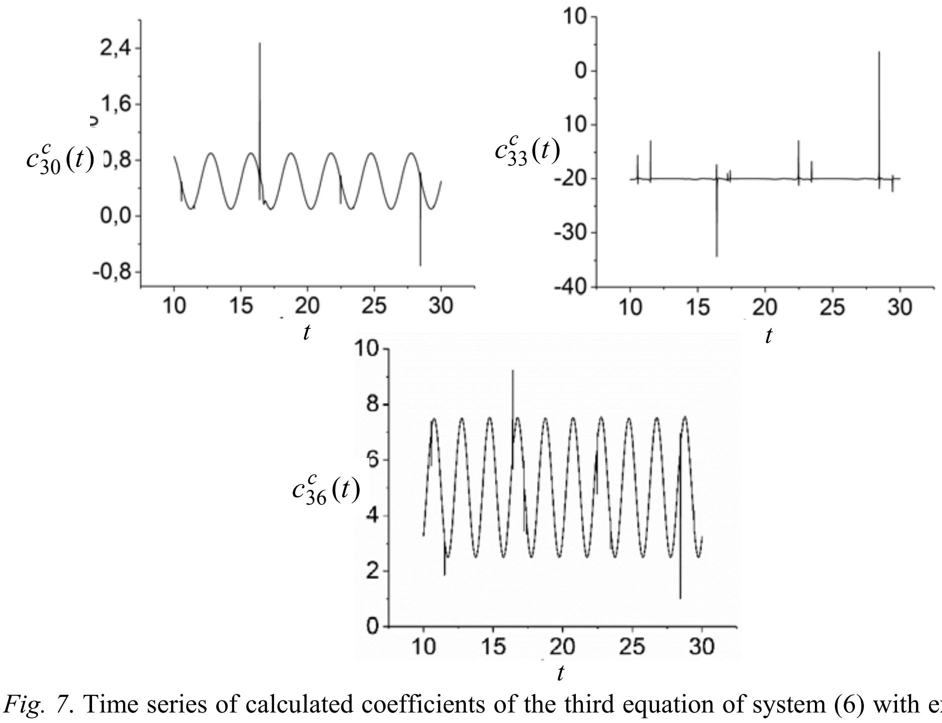

Fig. 7. Time series of calculated coefficients of the third equation of system (6) with external actions (8). Calculation was performed using simplified structure of the equation

If the data in Table 2 are not informative enough to confidently determine periods of external actions, one can repeat the numerical experiment on a larger time interval or/and use the results for more than five least δ values when creating the table. Such a numerical experiment was carried out for system (6) with external actions (8), which was solved over an interval of $200s$. As a result of applying the algorithm, the values given in Table 3 were obtained.

As one can see, the data in Table 3 confirm the correctness of the relationship $\tau = T = 6s$ obtained as a result of the analysis of the data in Table 2. The values of $\tau = 12s$ and $\tau = 18s$ in Table 3 are additional arguments for such a conclusion, since these values are obviously the multiples of $\tau = 6s$.

Table 3. The same as in Table 2, for system (6) with external actions (8). We used a time interval of $200s$ and ten τ values for which $\delta \to \min$

№	**The τ values calculated for the coefficients of equation (7) at $\delta \to \min$**									
	$c_{30}(t)$	$c_{31}(t)$	$c_{32}(t)$	$c_{33}(t)$	$c_{34}(t)$	$c_{35}(t)$	$c_{36}(t)$	$c_{37}(t)$	$c_{38}(t)$	$c_{39}(t)$
1	9.81	2.91	2.00	16.51	**6.00**	**18.00**	6.32	**6.00**	14.13	2.90
2	2.69	**6.00**	8.03	10.02	**18.00**	**6.00**	5.94	**18.00**	6.36	19.48
3	9.48	**18.00**	7.62	7.29	1.36	2.70	7.99	**12.00**	16.95	17.79
4	16.82	4.89	8.00	8.18	8.05	17.06	6.55	9.98	**12.00**	1.86
5	9.68	17.29	18.74	11.40	12.31	**12.00**	12.90	14.05	9.99	14.91
6	8.91	9.12	**18.00**	19.25	2.55	14.14	12.89	2.59	17.45	4.49
7	1.96	2.90	**6.00**	15.50	10.11	1.43	10.88	1.47	14.97	15.00
8	11.29	15.96	**12.00**	13.56	1.02	14.18	18.50	12.16	**6.00**	16.69
9	5.20	1.37	7.33	15.70	19.09	18.68	19.65	16.21	**18.00**	**6.00**
10	2.87	9.72	6.72	12.50	19.39	17.94	12.32	6.01	17.98	3.00

IDENTIFICATION OF THE EQUATION UNDER NON-SINUSOIDAL PERIODIC EXTERNAL ACTION

In this section we investigate system (6) with such coefficients:

$$d = 0.15, \;\; c_{30}(t) = 0.5, \;\; c_{33}(t) = -20,$$

$$c_{36}(t) = 5 + 2\sin\left(\frac{2\pi t}{T} - \frac{\pi}{2}\right) + 1.25\sin\left(\frac{4\pi t}{T}\right) + 0.8\sin\left(\frac{8\pi t}{T} + \frac{\pi}{2}\right), \;\; T = 2s. \quad (9)$$

System (6) with coefficients (9) was solved on the interval of $100s$ with a step $\Delta t = 0.01s$. Fig. 8 shows the time series of system under study and its external action $c_{36}(t)$. Fig. 9 shows the phase trajectories of this system.

Our goal was to identify the third equation of system (6) with coefficients (9). For this purpose, the general structure of an equation of the form (7) was used. As a result of applying the algorithm, the data presented in Table 4 were obtained. As can be seen from this table, the smallest possible value of the period is $T = 2s$.

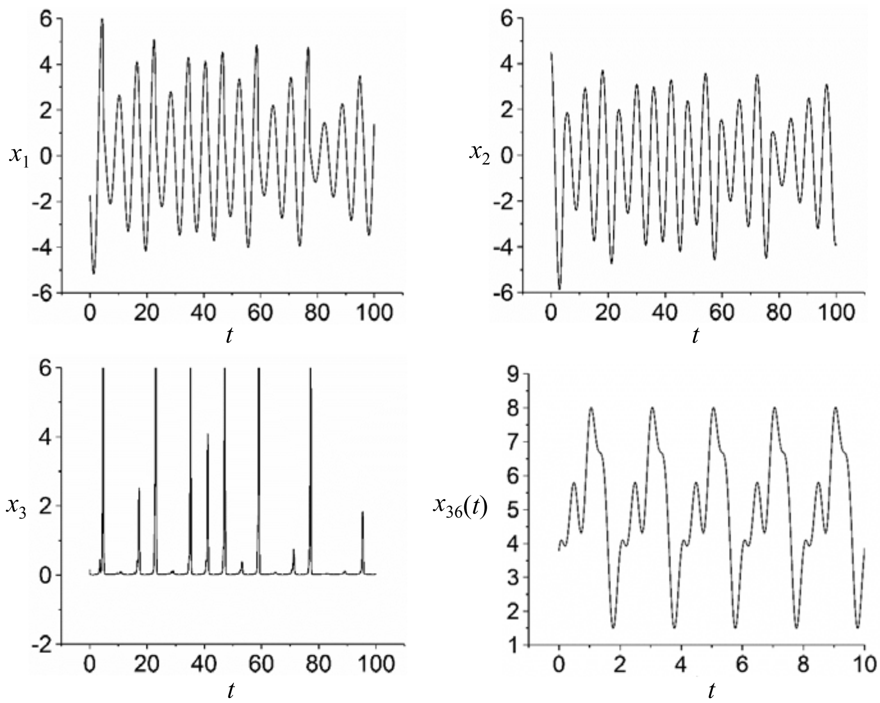

Fig. 8. Time series of system (6) with coefficients (9) and its external action $c_{36}(t)$

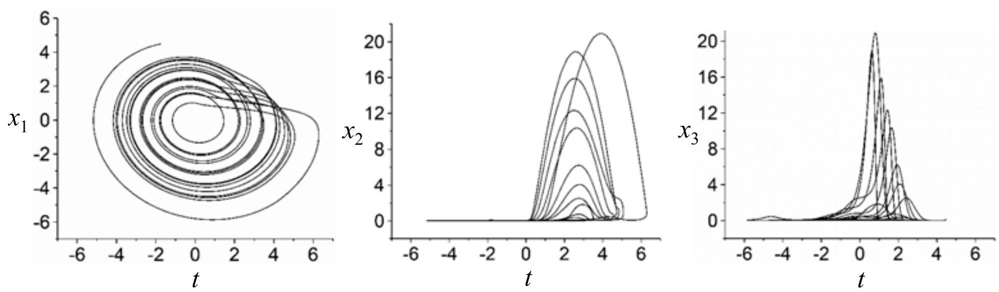

Fig. 9. Phase trajectories of system (6) with coefficients (9)

Fig. 10 shows the calculated time series of coefficients of different types: constant non-zero $c_{30}^{c}(t)$, $c_{33}^{c}(t)$, variable $c_{36}^{c}(t)$ and constant zero $c_{31}^{c}(t)$. The rest calculated coefficients of this equation have time series similar to $c_{31}^{c}(t)$, that is, they are zero.

T a b l e 4 . The same as in Table 3, for system (6) with coefficients (9)

№	The τ values calculated for the coefficients of equation (7) at $\delta \to \min$									
	$c_{30}(t)$	$c_{31}(t)$	$c_{32}(t)$	$c_{33}(t)$	$c_{34}(t)$	$c_{35}(t)$	$c_{36}(t)$	$c_{37}(t)$	$c_{38}(t)$	$c_{39}(t)$
1	**8.00**	**4.00**	1.49	**8.00**	**10.00**	**10.00**	2.11	**10.00**	**8.00**	10.57
2	9.23	**2.00**	**10.00**	**10.00**	**8.00**	**8.00**	2.75	**8.00**	**10.00**	7.56
3	**10.00**	9.24	**8.00**	6.62	**4.00**	4.54	2.77	**2.00**	**2.00**	5.78
4	10.31	**10.00**	10.67	5.67	**2.00**	**2.00**	8.71	5.60	**4.00**	4.50
5	**4.00**	**8.00**	**2.00**	**4.00**	10.97	**4.00**	1.63	**4.00**	7.60	**2.00**

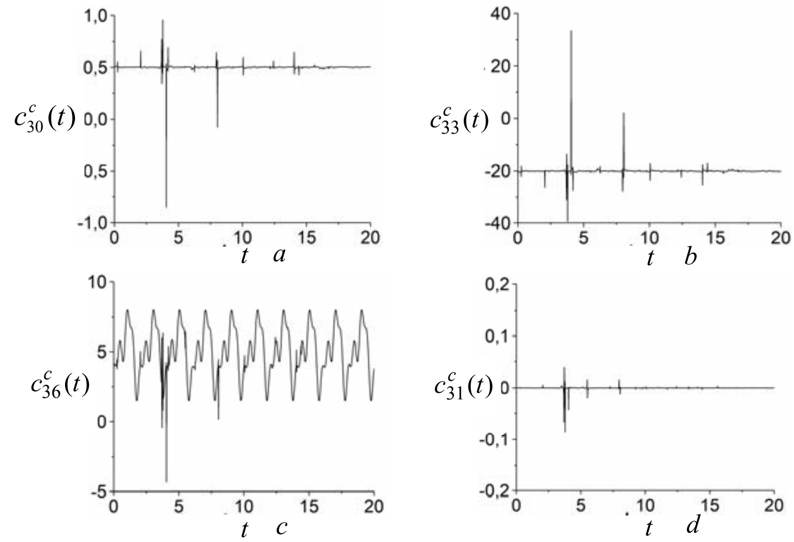

Fig. 10. Time series of calculated coefficients of the third equation of system (6) with coefficients (9)

After simplifying the structure, re-identification was carried out and the time series presented in Fig. 11 were obtained. Fig. 11, *d* shows the calculated time series of external action $c_{36}^{c}(t)$ (line 1) and the original one $c_{36}(t)$ (line 2). As we can see, these time series practically coincide, with the exception of points with singularity.

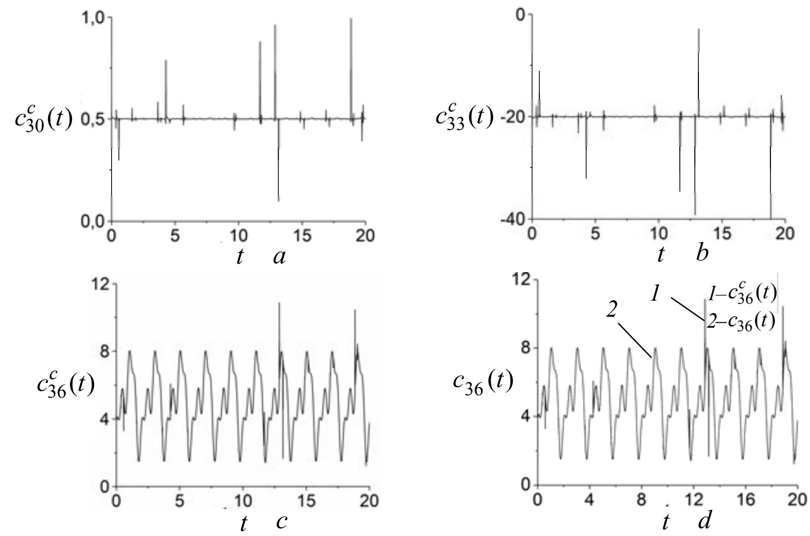

Fig. 11. Time series of calculated coefficients of the third equation of system (6) with coefficients (9). Calculation was performed using simplified structure of the equation

CONCLUSION

This study examines a number of special cases of using the proposed method for identifying nonlinear oscillatory systems with external periodic actions. The most complex part of the identification problem in this case is finding the periods of external actions. The first part of the algorithm is devoted to solving this problem. It can be noted that this method makes it relatively easy to find periods of external actions when identifying systems with an integer value of the ratio T_1/T_2 or

T_2/T_1 . If one of these conditions is met, estimating the values of T_1 and T_2 using this algorithm does not differ from the case $T_1 = T_2$ considered in [7].

With a fractional ratio T_1/T_2 or T_2/T_1 , much longer observations of system's functioning may be required in order to make an estimation.

The study of the system with non-sinusoidal periodic external action demonstrates that the proposed method is as effective as in the case of sinusoidal action.

A possible prospect for further development of the method could be, for example, a study of the dependence of the magnitude of the algorithm error on various parameters of the identified equations. It is also of interest to assess the influence of noise on the result of applying the algorithm.

REFERENCES

1. P.E. Kloeden, M. Rasmussen, "Nonautonomous dynamical systems," *Mathematical Surveys and Monographs*, vol. 176. Providence, Rhode Island: American Mathematical Society, 2011. doi: 10.1090/surv/176
2. P.E. Kloeden, C. Potzsche, "Nonautonomous dynamical systems in the life sciences," *Lecture Notes in Mathematics*, vol. 2102, pp. 3–39, 2013. doi: 10.1007/978-3-319-03080-7_1
3. A. Tarantola, *Inverse problem theory and methods for model parameter estimation*. Philadelphia: Society for Industrial and Applied Mathematics, 2005. doi: 10.1137/1.9780898717921
4. B.P. Bezruchko, D.A. Smirnov, "Constructing nonautonomous differential equations from a time series," *Phys. Rev. E.*, vol. 63, art. no. 016207, 2001. doi: 10.1103/PhysRevE.63.016207
5. T. Sauer, "Observing periodically forced systems of difference equations," *J. Difference Equ. Appl.*, vol. 16, no. 2-3, pp. 269–273, 2010. doi: 10.1080/10236190902870439
6. V.G. Gorodetskyi, "Identification of nonlinear systems with additive external action," *J. Aut. & Inf. Sci.*, vol. 50, no. 4, pp. 13–24, 2018. doi: 10.1615/JAutomatInfScien.v50.i4.20
7. V.G. Gorodetskyi, "Identification of nonlinear systems with periodic external actions (Part 1)," *System Research and Information Technologies*, no. 3, pp. 93–106, 2024. doi: 10.20535/SRIT.2308-8893.2024.3.06
8. O.E. Rössler, "An equation for continuous chaos," *Phys. Lett. A.*, vol. 57, no. 5, pp. 397–398, 1976. doi: 10.1016/0375-9601(76)90101-8

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INFORMATION ON THE ARTICLE

Viktor G. Gorodetskyi, ORCID: 0000-0003-4642-3060, National Technical Universityof Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Ukraine, e-mail: v.gorodetskyi@ukr.net

ІДЕНТИФІКАЦІЯ НЕЛІНІЙНИХ СИСТЕМ З ПЕРІОДИЧНИМИ ЗОВНІШНІМИ ДІЯМИ (Частина ІІ) / В.Г. Городецький

Анотація. Подано результати дослідження, яке є продовженням попередніх досліджень автора. Розглянуто більш складні задачі ідентифікації нелінійних систем з періодичними зовнішніми впливами. Показано, що запропонований раніше метод також застосовний, коли періоди зовнішніх дій в одному диференціальному рівнянні можуть відрізнятися. При цьому співвідношення між значеннями періодів може бути як цілим, так і дробовим. Сформульовано умови, за яких це можливо, і які базуються на теоремі, доведеній у попередній роботі. Частину цього дослідження присвячено проблемі ідентифікації хаотичної системи з вхідною несинусоїдальною дією. Для створення такої зовнішньої дії використано функцію з трьома гармонічними складовими. Чисельний експеримент підтвердив ефективність алгоритму і в цьому випадку.

Ключові слова: ідентифікація, звичайне диференціальне рівняння, зовнішня дія, періодичний коефіцієнт, сталий коефіцієнт.