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NEW APPROACH TO FINDING EIGENVECTORS FOR REPEATED EIGENVALUES OF A MATRIX

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Abstract. An efficient method of calculating eigenvectors for multiple eigenvalues of a matrix is proposed. This method is based on a formalized transformation of the problem of solving degenerate systems of equations into a regular problem by "repairing" their matrices and correspondingly correcting the right-hand sides of the equations, as well as "exclusion" during calculations from the spectrum eigenvalues of the matrix of one of the multiple values. In the case of non-defective multiples of the matrix, orthogonal eigenvectors are formed in contrast to the results obtained using the Mathematica program.

Keywords: eigenvectors, multiples of eigenvalues, algebraic and geometric multiplicity, solutions of degenerate systems, change of spectrum of a matrix, defective and non-defective multiples of a matrix.

INTRODUCTION

Finding the eigenvectors x_i for multiple eigenvalues λ_i of the matrix A is the least formalized task of modern Linear Algebra, as it is related to the solution of homogeneous (degenerate) systems of equations that have an infinite number of solutions:

$$B_i x_i = (A - \lambda_i E) x_i = 0 \quad \text{for} \quad i = 1, \dots, n, \tag{1}$$

since by definition the eigenvectors cannot be zero even for zero eigenvalues. An eigenvector corresponding to an eigenvalue creates an eigenspace associated with it. The set of all eigenvectors for different eigenvalues forms the vector space of the matrix.

It is well known that equation (1) has nonzero solutions for the vector x_i if and only if the matrix $(A - \lambda_i E)$ has a zero determinant, which determines the characteristic polynomial of the matrix [1–3]. It can be used to find the eigenvalues of the matrix (for small order tasks) together with a more powerful QR algorithm with orthogonal Householder or Givens rotation matrices, which reduce the original matrix to a triangular matrix, on the diagonal of which there are real eigenvalues or 2×2 blocks of eigenvalues (for large-scale tasks).

The article attempts to formalize the procedure for solving equation (1), excluding the traditional manual selection of individual components of the solution vector x_i in order to eliminate the degeneracy of the problem. Special attention is paid to the case of multiple eigenvalues, for which equation (1) has the same form, but there can be both different and coincident solutions. In the latter case, the rank of the matrix, which is determined by the number of independent eigenvectors, is lower than its order, so such multiples of eigenvalues are called "defective" [4–6].

The analysis of publications showed that there is great uncertainty in the issue of finding eigenvectors for multiples of the matrix, the Internet is full of requests from specialists of different countries for help and consultations [7–11] and educational materials [12–14]. This motivated the conduct of own research, the results of which formed the basis of this article.

STATE OF AFFAIRS

The existing method of solving the problem of finding eigenvectors x_i for multiple eigenvalues λ_i of the matrix A is best considered using some examples, say, the matrix

$$\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \tag{2}$$

which has a spectrum of eigenvalues $\lambda_1 = 10$ and $\lambda_2 = \lambda_3 = 1$.

When choosing an eigenvalue $\lambda_2 = 1$, the degenerate system of equations (1) is reduced to the form that demonstrates the relationship of all three equations:

$$B_2 x_2 = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3}$$

The system of equations (2) is transformed into the following form by the "row-reduction" procedure (i.e., reduction to a normal trapezoidal form):

$$\begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{4}$$

from which the following expression is formed

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_2 - 0.5v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} -0.5 \\ 0 \\ 1 \end{bmatrix}.$$

Then the eigenvectors corresponding to the eigenvalues $\lambda_2 = \lambda_3 = 1$ have the form:

$$x_2 = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -0.5 \\ 0 \\ 1 \end{bmatrix}, \quad |\alpha| + |\beta| \neq 0. \tag{5}$$

The condition $|\alpha| + |\beta| \neq 0$ excludes the selection of a zero eigenvector. At the same time, α and β can take any values, since the degenerate equations (3) have many solutions. By choosing different α and β, using (5), different values of x_2 are obtained and checked whether they satisfy the basic equation

$$Ax_i = \lambda_i x_i . \qquad (6)$$

For example, choosing, say, $\alpha = 0$ and $\beta = 1$, we get an eigenvector

$$x_2 = \{-0.5, 0, 1\} , \qquad (7)$$

which satisfies (6). On the contrary, choosing $\alpha = 1$ and $\beta = 0$, we get a solution

$$x_3 = \{-1, 1, 0\} , \qquad (8)$$

which also satisfies the basic equation (6).

It is interesting that when choosing $\alpha = 1$ and $\beta = 1$, we get according to (5), an eigenvector

$$x_4 = \{-1.5,\ 1,\ 1\} ,$$

which also satisfies equation (6) and is independent with respect to vectors x_2 and x_3. But they are not all orthogonal because $x_2 x_3 = 0.5$, $x_3 x_4 = 2.5$ and $x_2 x_4 = 1.75$.

In the case of multiple eigenvalues of the matrix, two of them are chosen from the set of independent solutions obtainable from (5) (eg, x_2 and x_3) as the corresponding eigenvectors for the multiple eigenvalues. It is clear that when the selected eigenvectors are multiplied by an arbitrary scale coefficient *m*, the basic equation (6) continues to be fulfilled.

For the completeness of the picture, one more method should be mentioned, which recommends, in the case of repeated eigenvalues, to use instead of equation (1) its modification [1, 2]

$$B_i^k x_i = (A - \lambda_i E)^k x_i = 0 ,$$

where k is an indicator of the algebraic multiplicity of an eigenvalue, and the set of solutions x_i for $k > 1$ corresponds to the so-called root eigenvectors. But, based on equation (3), it can be shown that this is rather a delusion. Indeed, for $k = 2$ we get instead of (3) the expression:

$$B_2^2 x_2 = \begin{bmatrix} 36 & 36 & 18 \\ 36 & 36 & 18 \\ 18 & 18 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which is transformed by the “row-reduction” procedure to the already known equation (4)

$$\begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

what indicates that the eigenvectors for multiple eigenvalues can be chosen from the set of solutions of equation (4), as was done above. A similar result is maintained if the multiplicity index k is increased.

Most likely, the considered selection of solutions is implemented in the well-known Mathematica program (the algorithmic support of which, unfortunately, is not described in detail in its documentation), because with its help (Fig. 1) for multiples of $\lambda_2 = \lambda_3 = 1$ of the matrix (2) eigenvectors $x_2 = \{-1, 0, 2\}$, $x_3 = \{-1, 1, 0\}$ can be obtained, which coincide with the accuracy of the coefficient with the values (7) and (8), which were previously manually selected when solving the system (4).

```
A={{5,4,2},{4,5,2},{2,2,2}};
{vals, vecs} = Eigensystem[A]
{{10, 1, 1}, {{2, 2, 1}, {-1, 0, 2}, {-1, 1, 0}}
```

Fig. 1. A fragment of the Mathematica code

It is interesting to compare the results of calculations obtained with the help of the Mathematica for two matrices that have the same spectrum of eigenvalues, but the multiples of the second one are defective (Fig. 2).

a	*b*
A1={{3,0,1},{0,3,2},{0,0,1}}; {vals, vecs} = Eigensystem[A1] {{**3, 3**, 1}, {{**0, 1, 0**}, {**1, 0, 0**}, {-1, -2, 2}}}	A2={{3,1,1},{0 ,3,2}, {0,0,1}}; {vals, vecs} = Eigensystem[A2] {{**3, 3, 1**}, {{**1, 0, 0**}, {**0, 0, 0**}, {0, -1, 1}}}

Fig. 2. Eigenvectors of two matrices with the same eigenvalues

The defect of a multiple eigenvalues matrix A2 is reflected in the Mathematica results by generation of a **zero eigenvector** (Fig. 2, *b*), what can mislead beginners who suspect an error in the program's operation.

But the Mathematica, unfortunately, sometimes contradicts itself, because it is enough to use the another its operator *JordanDecomposition*[A], related also to the calculation of eigenvectors and eigenvalues, and to find out with surprise that the same matrix A2 now has different eigenvectors $x_2 = \{1, 0, 0\}$ and $x_3 = \{0, 1, 0\}$ for the same multiple $\lambda_2 = \lambda_3 = 3$ (Fig. 3).

```
A2={{3,1,1},{0 ,3,2}, {0,0,1}};
JordanDecomposition [A2]
{{{0, 1, 0}, {-1, 0, 1}, {1, 0, 0}}, {{1, 0, 0}, {0, 3, 1}, {0, 0, 3}}}
```

$$\text{/MatrixForm} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

Fig. 3. The Jordanian normal form of the matrix A2

But the obtained value of x_2 does not satisfy the basic equation (6). In addition, the eigenvector $x_1 = \{0, 1, 0\}$ for $\lambda_1 = 1$ differs from its value $x_1 = \{0, -1, 1\}$ shown before in Fig. 2, and also does not satisfy equation (6).

Since the algorithmic core of the Mathematica is also used in other well-known calculation programs (Matlab, Mathcad, Maple), the results of their application to calculating eigenvectors for multiple eigenvalues will be similar.

THE PROPOSED METHOD

The paper contains a procedure for generating **orthogonal vectors** for multiple non-defective eigenvalues, what does not interfere with Mathematica, and two depended vectors for defective multiples. In this case, the solution of degenerate systems of type (3) is formalized by diagonal correction of the systems matrix after "row-reduction" (4) with a simultaneous correction of the zero vector of the right side of the system.

The system's degeneration (4) is manifested by zero k-th diagonal elements in its matrix. Similar to the method of diagonal correction [3], this matrix is "repaired" (so that degeneracy is eliminated) by replacing zero diagonal elements with a number equal to one or by some constant g, which is chosen to be equal to the middle value of the elements of the matrix B row. Then the solution is ongoing with the already ingenerated matrix and the new right-hand side $b_1 = \{0,\dots 1,\dots 0\}$, represented by a transposed vector of dimension n, such that all elements are zero and only k-th elements are equal to one.

However, if at the same time there is a zero column and row in the matrix of equations (4), then only the position of the diagonal element of the column is adjusted in the vector of the right part.

Let us illustrate what has been said with the example of the degenerate system of equations (4):

$$\begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \qquad (9)$$

The zero second and third diagonal elements of the zero rows of the original matrix are corrected by introducing the constant $g_2 = g_3 = 1$ and an additional vector of the right part b_2 is formed. From the solution of the adjusted system, we get

$$x_2 = \{-0.5, 0, 1\} \qquad (10)$$

or normalized value

$$X2 = Normalize[\mathrm{N}[x_2]] = \{-0.447214, 0., 0.894427\}. \qquad (11)$$

The eigenvector x_1 of the matrix for $\lambda_1 = 10$ is found quite similarly. In this case, the system of equations (1) looks like:

$$B_1 x_1 = \begin{bmatrix} -5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and by the "row-reduction" procedure it is transformed into the following form:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (12)$$

Unlike the matrix of the system of equations (4), the matrix of the system (12) has only one zero row, so its correction is performed differently:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (13)$$

As a result, we obtain the solution of the adjusted system (13)

$$x_1 = \{2, 2, 1\} \quad (14)$$

or normalized value

$$X1 = Normalize\,[N[x_1]] = \{0.666667,\ 0.666667,\ 0.333333\}. \quad (15)$$

Let us now consider an innovative procedure for finding the eigenvector for the second multiple of the eigenvalue $\lambda_2 = 1$ of the matrix. For this purpose, it is proposed to apply the following transformation of the matrix A, in which one of its multiple roots is excluded (zeroed), and then the problem is reduced to the previous one, when all the eigenvalues of the new matrix A1 are different. Such a transformation is performed according to the formula [3]:

$$\mathrm{A1} = A - \lambda_2 X_2 \otimes X_2 = A - \lambda_2 \cdot \mathrm{KroneckerProduct}\left[X_2, X_2\right], \quad (16)$$

where vector multiplication according to Kronecker, which results in a matrix, and the normalized eigenvector $X2$ (11) are used.

According to the formula (16) taking into account (11), we build the matrix

$$\mathrm{A1} = \begin{bmatrix} 4.8 & 4 & 2.4 \\ 4 & 5 & 2 \\ 2.4 & 2 & 1.2 \end{bmatrix},$$

for which the spectrum of eigenvalues $\lambda = \{10., 1., 3.8613110^{-16}\}$ does not contain multiples.

According to (1), we obtain a homogeneous system of equations

$$B_3 x_3 = \begin{bmatrix} 3.8 & 4 & 2.4 \\ 4 & 4 & 2 \\ 2.4 & 2 & 0.2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (17)$$

Using the “row-reduction” procedure, the system of equations (17) is transformed and then is being “repaired” taking into account the fact that after the “row-reduction” procedure, only one zero row is formed in the matrix:

$$B_3 x_3 =$$

$$= \begin{bmatrix} 3.8 & 4 & 2.4 \\ 4 & 4 & 2 \\ 2.4 & 2 & 0.2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2.5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (18)$$

As a result of the solution (18), we obtain the value of the second eigenvector

$$x_3 = \{2, -2.5, 1\} \tag{19}$$

or in normalized form

$$X3 = Normalize[N[x_3]] = \{0.596285, -0.745356, 0.298142\} \tag{20}$$

Thus, for the matrix (2) the following Eigensystem[A] is obtained by the proposed method

$$\{vals, vecs\} = \{\{10,1,1\}, \{\{2,2,1\}, \{-0.5,0,1\}, \{\mathbf{2}, \mathbf{-2.5}, \mathbf{1}\}\} \tag{21}$$

which differs from the results of Mathematica

$$\{vals, vecs\} = \{\{10,1,1\}, \{\{2,2,1\}, \{-1,0,2\}, \{\mathbf{-1}, \mathbf{1}, \mathbf{0}\}\} \tag{22}$$

presented in Fig. 1, by the value of the eigenvector for the second multiple eigenvalue $\lambda_3 = 1$.

The obtained eigenvectors given (10), (14) and (19) are **orthogona**l, since

$$x_1 x_3 = 0, \ x_1 x_2 = 0 \ \text{ and } \ x_2 x_3 = 0 .$$

If in the Mathematica's solution (22) we denote different components as $y_1 = \{2,2,1\}$, $y_2 = \{-1,0,2\}$ and $y_3 = \{-1,1,0\}$, then we can make sure, what

$$y_1 y_3 = 0, \ y_1 y_2 = 0, \ \text{but} \ \ y_2 y_3 = 1 .$$

This means that these vectors y_i although they satisfy the corresponding basic equations (6), are **not orthogonal** and therefore, unlike the set of eigenvectors x_i from (21), cannot ensure unmistakably the canonical *JordanDecomposition*[A] operation for the matrix *A*, when

$$A = TDT^t , \tag{23}$$

where T is the orthogonal matrix of eigenvectors, and *D* is the diagonal matrix of all eigenvalues, including multiples. Indeed, using the normalized values of the obtained eigenvectors X1, X2 and X3 from the corresponding formulas (15), (11) and (20), it is possible to build

$$T = \begin{bmatrix} 0.666667 & -0.447214 & 0.596285 \\ 0.666667 & 0 & -0.745356 \\ 0.333333 & 0.894427 & 0.298142 \end{bmatrix}, \quad D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and make sure that according to (23)

$$TDT^t = \{\{5.00001, 4., 2.\}, \{4., 5., 2.\}, \{2., 2., 2.\}\} = A .$$

For comparison, if you normalize the eigenvectors of the matrix *A* obtained with the help of Mathematica (22), you can get:

$$Y1 = Normalize[N[y_1]] = \{0.666667, 0.666667, 0.333333\},$$

$$Y2 = Normalize[N[y_2]] = \{-0.447214, 0., 0.894427\},$$

$$Y3 = Normalize[N[y_3]] = \{-0.707107, 0.707107, 0.\} .$$

and instead of the orthogonal matrix T construct another matrix

$$T_1=\begin{bmatrix}0.666667 & -0.447214 & -0.707107\\ 0.666667 & 0 & -0.707107\\ 0.333333 & 0.894427 & 0\end{bmatrix},$$

with which we can check whether equation (23) is satisfied:

$$A^*=T_1DT_1^t=\begin{bmatrix}5.14445 & 3.94445 & 1.82222\\ 3.94445 & 4.94445 & 2.22222\\ 1.82222 & 2.22222 & 1.91111\end{bmatrix}\neq A, \ .$$

while $\mathrm{Det}[T_1]=\mathrm{Det}[T_1^t]=-0.948684$ and $\mathrm{Det}[A^*]=9.00001$ instead of 10.

The same erroneous Mathematica's result may be obtained by applying the standard JordanDecomposition[A] operator.

The obtained result calls into question the existing lemma that orthogonal eigenvectors correspond only to different simple eigenvalues [14], which was formulated, most likely, on the basis of practical results obtained with the help of the traditional selection of solutions of a homogeneous system of equations, considered above using the example of the system (3). A new approach with the exclusion of multiples and consideration of two homogeneous systems of equations provides new opportunities.

It seems interesting, using the method described above, to find the eigenvectors of the matrix A2 for its eigenvalue's spectrum $\lambda=\{3,3,1\}$ for which the Mathematica generates a solution with a *zero eigenvector* (Fig. 2).

For the first multiple eigenvalue $\lambda_1=3$, by analogy with the above example, instead of equation (9), we obtain the following expression

$$\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_1\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\Rightarrow\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix},$$

from which we find $x_1=\{1,0,0\}$.

Using the obtained value of x_1, which coincides with its normalized value of X1, to exclude, according to (16), one of the multiples of the matrix A2, we find the matrix A3

$$A3=\begin{bmatrix}0 & 1 & 1\\ 0 & 3 & 2\\ 0 & 0 & 1\end{bmatrix},$$

which has a modified spectrum of eigenvalues $\lambda=\{0,3,1\}$ and for which, by analogy with (18), we construct an equation for finding the eigenvector of the second multiple $\lambda_2=3$ of the matrix A2:

$$\begin{bmatrix}1 & -1/3 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\Rightarrow\begin{bmatrix}1 & -1/3 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix},$$

from which we get $x_2 = \{1, 0, 0\}$.

As you can see, the values of x_1 and x_2 are linearly dependent (they just coincide), which indicates a defect in the multiple eigenvalues of the matrix.

CONCLUSIONS

One of the most important tasks of computational mathematics is the creation of effective and stable algorithms for finding the eigenvalues and vectors of a matrix [1]. They are a powerful tool that provides a deep understanding of matrix properties and opens wide perspectives for its application. Possession of this tool opens up opportunities for research and innovation in various fields of science and technology (for example, for identifying the main components and clustering of data during their analysis, for filtering signals and extracting a useful signal, for clustering and pattern recognition, etc.).

The state of affairs with the formalization of finding the eigenvectors of a matrix in general and for multiple eigenvalues in particular requires better. The article takes a certain step in this direction and proposes an innovative method of calculating eigenvectors for multiple eigenvalues of a matrix, which is based on the formalized transformation of the problem of solving degenerate systems of equations into a regular problem by "repairing" their matrices and by correspondingly correcting the right-hand sides of the equations, as well as "exclusion" of one of the multiple values from the spectrum of eigenvalues of the matrix during calculations of eigenvectors for multiples eigenvalues. In the case of non-defective multiples eigenvalues of the matrix, this method allows you to form orthogonal eigenvectors in contrast to the results obtained using the Mathematica.

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НОВИЙ ПІДХІД ДО ПОШУКУ ВЛАСНИХ ВЕКТОРІВ ДЛЯ КРАТНИХ ВЛАСНИХ ЧИСЕЛ МАТРИЦІ / А.І. Петренко

Анотація. Запропоновано ефективний метод обчислення власних векторів для кратних власних чисел матриці, який базується на формалізованому перетворенню задачі розв'язання вироджених систем рівнянь у звичайну задачу шляхом «ремонту» їх матриць і відповідного корегування правих частин рівнянь, а також «вилучення» під час обчислень зі спектра власних чисел матриці одного з кратних значень. У випадку недефектних кратних чисел матриці формуються ортогональні власні вектори на відміну від результатів, отриманих за допомогою програми Mathematica.

Ключові слова: власні вектори, кратні власні числа, алгебрична і геометрична кратність, розв'язання вироджених систем, зміна спектра матриці, дефектні і недефектні кратні числа матриці.