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CLASSICAL SPECIAL FUNCTIONS OF MATRIX ARGUMENTS

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Abstract. This article focuses on a few of the most commonly used special functions and their key properties and defines an analytical approach to building their matrix-variate counterparts. To achieve this, we refrain from using any numerical approximation algorithms and instead rely on properties of matrices, the matrix exponential, and the Jordan normal form for matrix representation. We focus on the following functions: the Gamma function as an example of a univariate function with a large number of properties and applications; the Beta function to highlight the similarities and differences from adding a second variable to a matrix-variate function; and the Jacobi Theta function. We construct explicit function views and prove a few key properties for these functions. In the comparison section, we highlight and contrast other approaches that have been used in the past to tackle this problem.

Keywords: matrix, special function, matrix function, gamma function, beta function, Jacobi theta function, Jordan normal form.

INTRODUCTION

Data with matrix responses for each experiment are increasingly common in modern statistical problems. For example, observations over a time period can be viewed holistically as a matrix variable, labeling the rows and columns as time and actual measurements respectively. Temporal and spatial data, multivariate growth curve data, imaging data, and data from cross-sectional designs also generate matrix-valued responses. On the other hand, many of these phenomena are still often built on generalized cases of classical problems, many of which are solved, or at least interpreted or simplified, by special functions. Therefore, the motivation of the study was to combine these two parts and to do so as generally as possible analytically, without relying on a specific problem or purely numerical methods. There were earlier studies on this topic, but they were aimed at either generalizing a specific concept (Mitra S. 1970 [1]), or calculating values for certain classes of matrices needed for further calculations (Kishka Z., Saleem M. 2019 [2]). In this article, based on the theory of matrices, matrix exponents and using the Jordanian canonical form of matrices, we formulate a basic toolkit of definitions and key properties of special matrix-variate functions. These properties are applicable to the widest range of matrices and have an explicit form, that is, they can be used for further research with minimal changes.

First, the definition and key properties for the matrix Gamma function will be given, as an example of a univariate special function, followed by a series of two-variable special functions such as the Beta function and the Jacobi Theta function. A comparison of the obtained results with the existing methods mentioned above will also be made

UNIVARIATE SPECIAL FUNCTIONS

1 GAMMA FUNCTION

1.1 the definition and the general form of the matrix-variate gamma function

For the Gamma function and all subsequent special functions, the integral definition of functions was taken as the basis of the study. Specifically for the Gamma function and the Beta function, the following shift was also made to simplify the calculations:

Definition. For an arbitrary matrix A, we define

$$\Gamma(A) = \int_0^\infty x^{A-I} e^{-x} dx = \begin{matrix}\text{to simplify further}\\ \text{calculations replace } A - I = B\end{matrix} = \int_0^\infty x^B e^{-x} dx. \quad (2.1)$$

From this definition using the matrix, we get the following form for the matrix-variate Gamma function:

For an arbitrary matrix $A \in \mathbb{R}^{(k\times k)}$, which has the Jordanian canonical form $J = U^{-1}AU$ its' Gamma function will have the form:

$$\Gamma(A) = U\Gamma(J)U^{-1};$$

$$\Gamma(J) = \begin{pmatrix} \Gamma(J_{r_1}(\lambda_1)) & & & \\ & \Gamma(J_{r_2}(\lambda_2)) & & \\ & & \ddots & \\ & & & \Gamma(J_{r_l}(\lambda_l)) \end{pmatrix} \text{ — block matrix,}$$

where

$$\Gamma(J_{r_j}(\lambda_j)) =$$

$$= \begin{pmatrix} \Gamma(\lambda_j+1) & \frac{\Gamma'(\lambda_j+1)}{1!} & \frac{\Gamma''(\lambda_j+1)}{2!} & \cdots & \frac{\Gamma^{(r_j-2)}(\lambda_j+1)}{(r_j-2)!} & \frac{\Gamma^{(r_j-1)}(\lambda_j+1)}{(r_j-1)!} \\ 0 & \Gamma(\lambda_j+1) & \frac{\Gamma'(\lambda_j+1)}{1!} & & \cdots & \frac{\Gamma^{(r_j-2)}(\lambda_j+1)}{(r_j-2)!} \\ 0 & 0 & \Gamma(\lambda_j+1) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \frac{\Gamma'(\lambda_j+1)}{1!} & \frac{\Gamma''(\lambda_j+1))}{2!} \\ 0 & 0 & 0 & \cdots & \Gamma(\lambda_j+1) & \frac{\Gamma'(\lambda_j+1)}{1!} \\ 0 & 0 & 0 & \cdots & 0 & \Gamma(\lambda_j+1) \end{pmatrix} \quad (2.2)$$

The blocks of the resulting matrix correspond to the blocks of each of the eigenvalues of the Jordan matrix J of the matrix A and have the corresponding dimensions $(r_i \times r_i)$. It should also be noted that these matrices are upper-triangular, that is, they have zero-values below the main diagonal. This fact will also be important for the subsequent special functions.

1.2 Main functional equation

One of the most important and used properties of the Gamma function is the functional equation, as several other properties of the Gamma function are based on it. Also, this property allows you to recursively find the values of the function, thereby significantly simplifying calculations.

For a scalar argument, the identity has the following form:

$$\Gamma(\lambda+1)=\lambda*\Gamma(\lambda). \tag{2.3}$$

To prove this statement in the matrix case, we first consider the following auxiliary equality:

$$=\begin{pmatrix} \lambda+1 & 1 & 0 & \dots & 0 \\ 0 & \lambda+1 & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \lambda+1 & 1 \\ 0 & \dots & \dots & 0 & \lambda+1 \end{pmatrix}=J_{r_i}(\lambda+1). \tag{2.4}$$

Let us now use this and definition (2.2) to generalize identity (2.3):

$$\Gamma(J_{r_i}(\lambda_i)+I)\underset{(2.4)}{\overset{using}{=}}\Gamma\left(J_{r_i}(\lambda_i+1)\right)\underset{(2.2)}{\overset{using}{=}}$$

$$=\begin{pmatrix} \Gamma(\lambda_j+2) & \frac{\Gamma'(\lambda_j+1)}{1!} & \frac{\Gamma''(\lambda_j+1)}{2!} & \dots & \frac{\Gamma^{(r_j-2)}(\lambda_j+1)}{(r_j-2)!} & \frac{\Gamma^{(r_j-1)}(\lambda_j+2)}{(r_j-1)!} \\ 0 & \Gamma(\lambda_j+2) & \frac{\Gamma'(\lambda_j+2)}{1!} & & \dots & \frac{\Gamma^{(r_j-2)}(\lambda_j+1)}{(r_j-2)!} \\ 0 & 0 & \Gamma(\lambda_j+2) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \frac{\Gamma'(\lambda_j+2)}{1!} & \frac{\Gamma''(\lambda_j+2))}{2!} \\ 0 & 0 & 0 & \dots & \Gamma(\lambda_j+2) & \frac{\Gamma'(\lambda_j+2)}{1!} \\ 0 & 0 & 0 & \cdots & 0 & \Gamma(\lambda_j+2) \end{pmatrix}$$

Using the properties of the derivative of the Gamma function and the properties of the Gamma function itself:

$$\begin{pmatrix} \Gamma(\lambda_j+2) & \frac{\Gamma'(\lambda_j+1)}{1!} & \frac{\Gamma''(\lambda_j+1)}{2!} & \dots & \frac{\Gamma^{(r_j-2)}(\lambda_j+1)}{(r_j-2)!} & \frac{\Gamma^{(r_j-1)}(\lambda_j+2)}{(r_j-1)!} \\ 0 & \Gamma(\lambda_j+2) & \frac{\Gamma'(\lambda_j+2)}{1!} & & \dots & \frac{\Gamma^{(r_j-2)}(\lambda_j+1)}{(r_j-2)!} \\ 0 & 0 & \Gamma(\lambda_j+2) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \frac{\Gamma'(\lambda_j+2)}{1!} & \frac{\Gamma''(\lambda_j+2))}{2!} \\ 0 & 0 & 0 & \dots & \Gamma(\lambda_j+2) & \frac{\Gamma'(\lambda_j+2)}{1!} \\ 0 & 0 & 0 & \cdots & 0 & \Gamma(\lambda_j+2) \end{pmatrix}=$$

$$=\begin{pmatrix} (\lambda_i+1)\Gamma(\lambda_i+1) & \dfrac{(\lambda_i+1)\Gamma'(\lambda_i+1)+\Gamma(\lambda_i+1)}{1!} & \cdots & \cdots & \dfrac{(\lambda_i+1)\Gamma^{(r_i-1)}(\lambda_i+1)+(r_i-1)\Gamma^{(r_i-2)}(\lambda_i+1)}{(r_i-1)!} \\ 0 & (\lambda_i+1)\Gamma(\lambda_i+1) & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \dfrac{(\lambda_i+1)\Gamma'(\lambda_i+1)+\Gamma(\lambda_i+1)}{1!} \\ 0 & 0 & 0 & 0 & (\lambda_i+1)\Gamma(\lambda_i+1) \end{pmatrix}$$

Now we split the obtained matrix into two separate matrices, grouping all terms with the coefficient (λ_i+1) into the first, all others into the second:

$$\begin{pmatrix} (\lambda_i+1)\Gamma(\lambda_i+1) & \dfrac{(\lambda_i+1)\Gamma'(\lambda_i+1)}{1!} & \cdots & \dfrac{(\lambda_i+1)\Gamma^{(r_i-1)}(\lambda_i+1)}{(r_i-1)!} \\ 0 & (\lambda_i+1)\Gamma(\lambda_i+1) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \dfrac{(\lambda_i+1)\Gamma'(\lambda_i+1)}{1!} \\ 0 & 0 & 0 & (\lambda_i+1)\Gamma(\lambda_i+1) \end{pmatrix}+$$

$$+\begin{pmatrix} 0 & \dfrac{\Gamma((\lambda_i+1))}{1!} & \cdots & \dfrac{(r_i-1)\Gamma^{(r_i-2)}(\lambda_i+1)}{(r_i-1)!} \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \dfrac{\Gamma((\lambda_i+1))}{1!} \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

subtract (λ_i+1) from the first term as a matrix and reduce the factorial and coefficient in the second:

$$(\lambda_i+1)\begin{pmatrix} \Gamma(\lambda_i+1) & \dfrac{\Gamma'(\lambda_i+1)}{1!} & \cdots & \dfrac{\Gamma^{(r_i-1)}(\lambda_i+1)}{(r_i-1)!} \\ 0 & \Gamma(\lambda_i+1) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \dfrac{\Gamma'(\lambda_i+1)}{1!} \\ 0 & 0 & 0 & \Gamma(\lambda_i+1) \end{pmatrix}+$$

$$+\begin{pmatrix} 0 & \dfrac{\Gamma((\lambda_i+1))}{1!} & \dfrac{\Gamma''((\lambda_i+1))}{2!} & \cdots & \dfrac{\Gamma^{(r_i-2)}(\lambda_i+1)}{(r_i-2)!} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \dfrac{\Gamma((\lambda_i+1))}{1!} & \dfrac{\Gamma''((\lambda_i+1))}{2!} \\ \vdots & \vdots & \cdots & 0 & \dfrac{\Gamma((\lambda_i+1))}{1!} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}=$$

$$= (\lambda_i + 1)\Gamma(J_{r_i}(\lambda_i)) + \Gamma(J_{r_i}(\lambda_i))J_{r_i}(0).$$

That is, we get the following identity:

$$\Gamma(J_{r_i}(\lambda_i) + I) = (\lambda_i + 1)\Gamma(J_{r_i}(\lambda_i)) + \Gamma(J_{r_i}(\lambda_i))J_{r_i}(0).$$

Generalizing for the matrix $A = U\,J_{r_i}(\lambda_i)U^{-1}$:

$$\Gamma(A + I) = (\lambda_i + 1)\Gamma(A) + \Gamma(A)U\,J_{r_i}(0)U^{-1}. \qquad (2.5)$$

As we can see, it was possible to prove a property similar to (2.3), but to which the correcting term $\Gamma(A)U\,J_{r_i}(0)U^{-1}$ is added. Indeed, if we take the dimension of the matrix A as (1×1), i.e. return to a scalar variable, then the second term of the identity will be equal to 0 and we will return to the widely-known identity (2.3).

1.3 Euler's reflection formula

Before moving on to the generalization of the reflection formula, we give an additional auxiliary property:

$$\Gamma(mJ_r(\lambda_i)) = \int_0^\infty e^{mJ_r(\lambda_i)\ln x} e^{-x}dx = \int_0^\infty \begin{pmatrix} e^{m\lambda_i \ln x} & \cdots & \dfrac{(m\ln x)^{r_i-1} e^{m\lambda_i \ln x}}{(r_i-1)!} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{m\lambda_i \ln x} \end{pmatrix} e^{-x}dx =$$

$$= \begin{pmatrix} \Gamma(m\lambda_i + 1) & \dfrac{\Gamma'(m\lambda_i+1)}{1!} & \cdots & \dfrac{\Gamma^{(r_i-1)}(m\lambda_i+1)}{(r_i-1)!} \\ 0 & \Gamma(\lambda_i+1) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \dfrac{\Gamma'(m\lambda_i+1)}{1!} \\ 0 & 0 & 0 & \Gamma(m\lambda_i+1) \end{pmatrix}. \qquad (3.9)$$

Now let's return to Euler's reflection formula:

$$\Gamma(J_r(\lambda_i))\Gamma(I_{r_i} - J_r(\lambda_i)) =$$

$$= \left| I_{r_i} - J_r(\lambda_i) = \begin{pmatrix} 1-\lambda_i & -1 & 0 & \dots & 0 \\ 0 & 1-\lambda_i & -1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1-\lambda_i & -1 \\ 0 & \dots & \dots & 0 & 1-\lambda_i \end{pmatrix} = -J_{r_i}(\lambda_i - 1) \right| =$$

$$= \Gamma(J_{r_i}(\lambda_i))\Gamma(-J_{r_i}(\lambda_i+1)) \underset{(3.9)}{\overset{вл.}{=}} \begin{pmatrix} \Gamma(\lambda_i+1) & \dfrac{\Gamma'(\lambda_i+1)}{1!} & \cdots & \dfrac{\Gamma^{(r_i-1)}(\lambda_i+1)}{(r_i-1)!} \\ 0 & \Gamma(\lambda_i+1) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \dfrac{\Gamma'(\lambda_i+1)}{1!} \\ 0 & 0 & 0 & \Gamma(\lambda_i+1) \end{pmatrix} \times$$

$$\times\begin{pmatrix} \Gamma(-\lambda_i+2) & \dfrac{\Gamma'(-\lambda_i+2)}{1!} & \dots & \dfrac{\Gamma^{(r_i-1)}(-\lambda_i+2)}{(r_i-1)!} \\ 0 & \Gamma(-\lambda_i+2) & \dots & \vdots \\ \vdots & \vdots & \ddots & \dfrac{\Gamma'(-\lambda_i+2)}{1!} \\ 0 & 0 & 0 & \Gamma(-\lambda_i+2) \end{pmatrix}=$$

$$=\Gamma(J_{r_i}(\lambda_i))\Gamma(J_{r_i}(1-\lambda_i))$$

Derivation of Euler's reflection formula:

Consider the following product:

$$\Gamma\left(\frac{1}{2}I_{r_i}+J_{r_i}(\lambda_i)\right)\Gamma\left(\frac{1}{2}I_{r_i}-J_{r_i}(\lambda_i)\right);$$

$$\frac{1}{2}I_{r_i}+J_r(\lambda_i)=\begin{pmatrix} \frac{1}{2}+\lambda_i & 1 & 0 & \dots & 0 \\ 0 & \frac{1}{2}+\lambda_i & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \frac{1}{2}+\lambda_i & 1 \\ 0 & \dots & \dots & 0 & \frac{1}{2}+\lambda_i \end{pmatrix}=J_r\left(\lambda_i+\frac{1}{2}\right);$$

$$\frac{1}{2}I_{r_i}-J_r(\lambda_i)=\begin{pmatrix} \frac{1}{2}-\lambda_i & -1 & 0 & \dots & 0 \\ 0 & \frac{1}{2}-\lambda_i & -1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \frac{1}{2}-\lambda_i & -1 \\ 0 & \dots & \dots & 0 & \frac{1}{2}-\lambda_i \end{pmatrix}=-J_r\left(\lambda_i-\frac{1}{2}\right).$$

$$\Gamma\left(\frac{1}{2}I_{r_i}+J_{r_i}(\lambda_i)\right)\Gamma\left(\frac{1}{2}I_{r_i}-J_{r_i}(\lambda_i)\right)=\Gamma\left(J_r\left(\lambda_i+\frac{1}{2}\right)\right)\Gamma\left(-J_r\left(\lambda_i-\frac{1}{2}\right)\right)=$$

$$\begin{pmatrix} \Gamma\left(\lambda_i+\frac{3}{2}\right) & \dfrac{\Gamma'\left(\lambda_i+\frac{3}{2}\right)}{1!} & \dots & \dfrac{\Gamma^{(r_i-1)}\left(\lambda_i+\frac{3}{2}\right)}{(r_i-1)!} \\ 0 & \Gamma\left(\lambda_i+\frac{3}{2}\right) & \dots & \vdots \\ \vdots & \vdots & \ddots & \dfrac{\Gamma'\left(\lambda_i+\frac{3}{2}\right)}{1!} \\ 0 & 0 & 0 & \Gamma\left(\lambda_i+\frac{3}{2}\right) \end{pmatrix}\times$$

$$\times\begin{pmatrix} \Gamma\left(-\lambda_i+\frac{1}{2}\right) & \frac{\Gamma'\left(-\lambda_i+\frac{1}{2}\right)}{1!} & \dots & \frac{\Gamma^{(r_i-1)}\left(-\lambda_i+\frac{1}{2}\right)}{(r_i-1)!} \\ 0 & \Gamma\left(-\lambda_i+\frac{1}{2}\right) & \dots & \vdots \\ \vdots & \vdots & \ddots & \frac{\Gamma'\left(-\lambda_i+\frac{1}{2}\right)}{1!} \\ 0 & 0 & 0 & \Gamma\left(-\lambda_i+\frac{1}{2}\right) \end{pmatrix} = \Gamma(J_{r_i}(\lambda_i))\Gamma(J_{r_i}(1-\lambda_i)) =$$

$$= \begin{pmatrix} \Gamma\left(\lambda_i+\frac{3}{2}\right)\times \Gamma\left(-\lambda_i+\frac{1}{2}\right) & \dots & \dots & \swarrow\begin{pmatrix} i=0 \\ j=r-1 \end{pmatrix} \\ 0 & \Gamma\left(\lambda_i+\frac{3}{2}\right)\times \Gamma\left(-\lambda_i+\frac{1}{2}\right) & \sum_{k=0}^{j-i}\frac{\Gamma^{(k)}\left(\lambda_i+\frac{3}{2}\right)\Gamma^{(r-1-i-k)}\left(-\lambda_i+\frac{1}{2}\right)}{k!(r-1-i-k)!} & \vdots \\ \vdots & \ddots & \Gamma\left(\lambda_i+\frac{3}{2}\right)\times \Gamma\left(-\lambda_i+\frac{1}{2}\right) & \vdots \\ 0 & \dots & 0 & \Gamma\left(\lambda_i+\frac{3}{2}\right)\times \Gamma\left(-\lambda_i+\frac{1}{2}\right) \end{pmatrix}$$

2 BETA FUNCTION

2.1 Definition of the matrix-variate beta function

Similarly to the previous subsection, let's start with the definition of the matrix-variate Beta function, using the integral definition of the Beta function:

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

For two matrices $X,Y\in\mathbb{R}^{(k\times k)}$ with Jordan canonical forms $X=U_1J_1U_1^{-1}$; $Y=U_2J_2U_2^{-1}$ we consider the function of two matrix variables. We will first perform the following calculations for their Jordan matrices, respectively:

$$B(J_r(\lambda_1),J_r(\lambda_2))=\int_0^1 e^{\ln(t)J_r(\lambda_1)}e^{\ln(1-t)J_r(\lambda_2)}dt$$

Now we present and analyze the integral product separately:

$$e^{\ln(t)J_r(\lambda_1)}e^{\ln(1-t)J_r(\lambda_2)}=\begin{pmatrix} e^{\lambda_1\ln(t)}e^{\lambda_2\ln(1-t)} & \dots & M_{(l,m)} \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_1\ln(t)}e^{\lambda_2\ln(1-t)} \end{pmatrix},$$

where $M_{(l,m)}$ is an arbitrary product of the l-th row and m-th column of the initial matrices, and $m \ge l$:

$$M_{(l,m)}=\Sigma\left(0 \;\dots\; 0 \;\; e^{\lambda_1\ln(t)}e^{\lambda_2\ln(1-t)}\frac{(\ln(1-t))^{m-l}}{((m-l)!} \;\dots\; 0 \;\dots\; 0\right).$$

The number of zeros at the beginning and at the end of the product is l and $k-m-l$, respectively, so the resulting sum will consist of the middle part of the vector:

$$M_{(l,m)}=\sum_{j=0}^{m-l}\frac{(\ln(t))^j(\ln(1-t))^{(m-l)-j}}{j!((m-l)-j)!}e^{\lambda_1\ln(t)}e^{\lambda_2\ln(1-t)}.$$

Then, returning to the integral, we get the following:

$\int_0^1 e^{\lambda_1\ln(t)}e^{\lambda_2\ln(1-t)}dt=B(\lambda_1+1,\lambda_2+1)$ — elements on the main diagonal;

$$\int_0^1\sum_{j=0}^{m-l}\frac{(\ln(t))^j(\ln(1-t))^{(m-l)-j}}{j!((m-l)-j)!}e^{\lambda_1\ln(t)}e^{\lambda_2\ln(1-t)}=$$

$$=\begin{pmatrix} B(\lambda_1+1,\lambda_2+1) & \dots & \displaystyle\sum_{j=0}^{m-l}\frac{1}{j!((m-l)-j)!}\frac{\partial^{(m-l)}B(x,y)}{\partial x^j\partial y^{(m-l)-j}}\Bigg|_{\substack{x=\lambda_1+1\\ y=\lambda_2+1}} \\ \vdots & \ddots & \vdots \\ 0 & \dots & B((m-l)-j) \end{pmatrix}.$$

It should be noted that the resulting matrix, namely an arbitrary element in the form of the sum $M_{(l,m)}$ and the corresponding resulting sums of derivatives depend only on the difference of indices (l,m), and not on each of them separately. This, in turn, means that these elements are equal to each other for m and l on the corresponding diagonals, which significantly reduces the number of necessary calculations for finding the explicit form of the matrix $B(,)$ for specific values.

Now let us return to the initial general definition for arbitrary matrices X,Y:

$$B(X,Y)=\int_0^1 e^{\ln(t)X}e^{\ln(1-t)Y}dt=\begin{matrix}\text{using the Jordan}\\ \text{decomposition}\end{matrix}=$$

$$=\int_0^1 U_1e^{\ln(t)J_1}U_1^{-1}U_2e^{\ln(1-t)J_2}U_2^{-1}dt;$$

Compared to the situation with functions of one variable (e.g. Gamma function), when we start working with functions of several matrix variables, we have two different Jordan transformations, and that is, two different matrices U_1, U_2, which greatly complicates the task and makes it impossible to establish a direct relationship between $B(X,Y)$ and $B(J_1,J_2)$ to obtain a clear analytical view of the resulting matrix. In this regard, it is advisable to further consider the matrices X, Y as a pair of commuting matrices, which will give us the opportunity to find a common Jordan basis for them, i.e. $U_1 = U_2$. Also, in several points, it will allow the use of properties of the matrix exponent only for commuting matrices. Therefore, taking this into account, we get the following result:

$$B(X,Y) = \int_0^1 U e^{\ln(t)J_1} U^{-1} U e^{\ln(1-t)J_2} U^{-1} dt == U \int_0^1 e^{\ln(t)J_1} e^{\ln(1-t)J_2} dt U^{-1} =$$

$$= UB(J_1, J_2)U^{-1}.$$

2.2 Certain properties of the Beta function

1) Symmetry: $B(x, y) = B(y, x)$

Since commuting matrices were chosen for research from the previous point, symmetry for matrix arguments is also preserved.

2) Partial case of the function $B(1,x) = 1/x$.

For the matrix-variate function, let's start with $J_r(\lambda_1)$:

$$B(I, J_r) = \int_0^1 e^{\ln(1-t)J_r(\lambda_1)} dt$$

In this case, we have a single matrix of the form:

$$e^{\ln(1-t)J_r(\lambda_1)} = \begin{pmatrix} e^{\lambda_1 \ln(1-t)} & \dots & \dfrac{(\ln(1-t))^{k-1} e^{\lambda_1 \ln(1-t)}}{(k-1)!} \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_1 \ln(1-t)} \end{pmatrix};$$

Then, with the absence of a product, we go directly to the integral:

$$B(I, J_r(\lambda_1)) = \begin{pmatrix} B(1, \lambda_1 + 1) & \dots & \dfrac{1}{(k-1)!} \dfrac{\partial^{(k-1)} B(x,y)}{\partial y^{k-1}} \Bigg|_{\substack{x=1 \\ y=\lambda_1+1}} \\ \vdots & \ddots & \vdots \\ 0 & \dots & B(1, \lambda_1 + 1) \end{pmatrix} =$$

$$= \begin{pmatrix} B(1, \lambda_1 + 1) & \dots & \dfrac{1}{(k-1)!} \dfrac{\partial^{(k-1)} B(1,y)}{\partial y^{k-1}} \Bigg|_{y=\lambda_1+1} \\ \vdots & \ddots & \vdots \\ 0 & \dots & B(1, \lambda_1 + 1) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{\lambda_1 + 1} & \cdots & \frac{(-1)^{k-1}}{(k-1)!(\lambda_1+1)^{k-1}} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\lambda_1+1} \end{pmatrix} = \text{property of the jordan matrix} = f(J(\lambda_1 + 1)),$$

where $f(x) = \frac{1}{x}$.

So, we see a complete analogy with the property of the scalar Beta function. Summarizing:

$$B(I, X) = UB(I, J)U^{-1}.$$

2.3 Pascal's rule or the Beta function recurrence relation

Pascal's rule is one of the key identities in combinatorics and given the relationship between the Beta function and binomial coefficients, as well as its use for the recurrent computation of the Beta function, it will be appropriate to try to generalize it for two arbitrary commuting matrices.

$$B(X+I, Y) + B(X, Y+I) = \int_0^1 e^{\ln(t)(X+I)} e^{\ln(1-t)Y} dt +$$

$$+ \int_0^1 e^{\ln(t)X} e^{\ln(1-t)(Y+I)} dt = \int_0^1 (e^{\ln(t)c} e^{\ln(1-t)Y} + e^{\ln(t)X} e^{\ln(1-t)(Y+I)}) dt =$$

$$\int_0^1 (Ue^{\ln(t)J_1} U^{-1} e^{\ln(t)I} Ue^{\ln(1-t)J_2} U^{-1} + Ue^{\ln(t)J_1} U^{-1} Ue^{\ln(1-t)J_2} U^{-1} e^{\ln(1-t)I}) dt =$$

$$= \text{according to properties of commuting matrices and exponent} =$$

$$\int_0^1 (Ue^{\ln(t)J_1} e^{\ln(1-t)J_2} U^{-1} e^{\ln(t)I} + Ue^{\ln(t)J_1} e^{\ln(1-t)J_2} U^{-1} e^{\ln(1-t)I}) dt =$$

$$\int_0^1 (Ue^{\ln(t)J_1} e^{\ln(1-t)J_2} U^{-1} (e^{\ln(t)I} + e^{\ln(1-t)I})) dt .$$

According to the property of the matrix exponent, the two terms obtained are found as exponents of the diagonal matrix:

$$e^{\ln(t)I} + e^{\ln(1-t)I} = \begin{pmatrix} e^{\ln(t)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\ln(t)} \end{pmatrix} + \begin{pmatrix} e^{\ln(1-t)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\ln(1-t)} \end{pmatrix} = I.$$

So, the end result is as follows:

$$B(X+I, Y) + B(X, Y+I) = \int_0^1 (Ue^{\ln(t)J_1} e^{\ln(1-t)J_2} U^{-1} I) dt =$$

$$= U\int_0^1 e^{\ln(t)J_1} e^{\ln(1-t)J_2} dt U^{-1} = UB(J_1, J_2)U^{-1} = B(X,Y).$$

Similarly to the scalar Beta function, Pascal's rule holds and has the same form, unlike many other properties that have additional constructions when working with matrix variables.

3 JACOBI THETA FUNCTION

3.1 Definition of the main Jacobi Theta function in the form of an infinite sum

$$\vartheta(J_r(\lambda_1), J_r(\lambda_2)) = \sum_{n=-\infty}^{\infty} Q^{n^2} \mathrm{H}^n,$$

where $Q = e^{\pi i J_r(\lambda_2)}, \mathrm{H} = e^{2\pi i J_r(\lambda_1)}$.

As in the previous section, let's start with each of the factors separately and then move on to the general form of the product:

$$Q^{n^2} = e^{\pi i n^2 J_r(\lambda_2)} =$$

$$= \begin{pmatrix} e^{\pi i n^2 \lambda_2} & \frac{\pi i n^2 e^{\pi i n^2 \lambda_2}}{1!} & \cdots & \frac{(\pi i n^2)^{k-2} e^{\pi i n^2 \lambda_2}}{(k-2)!} & \frac{(\pi i n^2)^{k-1} e^{\pi i n^2 \lambda_2}}{(k-1)!} \\ 0 & e^{\pi i n^2 \lambda_2} & \cdots & \cdots & \frac{(\pi i n^2)^{k-2} e^{\pi i n^2 \lambda_2}}{(k-2)!} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & e^{\pi i n^2 \lambda_2} & \frac{\pi i n^2 e^{\pi i n^2 \lambda_2}}{1!} \\ 0 & 0 & \cdots & 0 & e^{\pi i n^2 \lambda_2} \end{pmatrix}$$

$$\mathrm{H}^n = e^{2\pi i n J_r(\lambda_1)} =$$

$$= \begin{pmatrix} e^{\pi i n \lambda_1} & \frac{\pi i n e^{\pi i n \lambda_1}}{1!} & \cdots & \frac{(\pi i n)^{k-2} e^{\pi i n \lambda_1}}{(k-2)!} & \frac{(\pi i n)^{k-1} e^{\pi i n \lambda_1}}{(k-1)!} \\ 0 & e^{\pi i n^2 \lambda_2} & \cdots & \cdots & \frac{(\pi i n)^{k-2} e^{\pi i n \lambda_1}}{(k-2)!} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & e^{\pi i n^2 \lambda_2} & \frac{\pi i n\, e^{\pi i n \lambda_1}}{1!} \\ 0 & 0 & \cdots & 0 & e^{\pi i n^2 \lambda_2} \end{pmatrix}.$$

Then the product of these matrices will be:

$$\left(Q^{n^2} \mathrm{H}^n\right) = \begin{pmatrix} e^{2\pi i n \lambda_1} e^{\pi i n^2 \lambda_2} & \cdots & M_{(l,m)} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{2\pi i n \lambda_1} e^{\pi i n^2 \lambda_2} \end{pmatrix},$$

where $M_{(l,m)}$ is the product of the l-th row and the m-th column of the initial matrices for $m \geq l$.

$$M_{(l,m)} = \Sigma\left(0\ldots0\, e^{2\pi in\lambda_1} e^{\pi in^2\lambda_2} \frac{(\pi in^2)^0}{0!}\frac{(2\pi in)^{m-l}}{(m-l)!}\ldots\frac{(\pi in^2)^{m-l}}{(m-l)!}\frac{(2\pi in)^0}{0!}0\ldots0\right),$$

while the number of zeros at the beginning and end will be equal to l and $k-m-1$ respectively. Then as a result we get the sum:

$$M_{(l,m)} = \sum_{j=0}^{m-l} \frac{(\pi in^2)^j (2\pi in)^{m-l-j}}{j!(m-l-j)!} e^{2\pi in\lambda_1} e^{\pi in^2\lambda_2}.$$

Thus, similar to the product for the Beta function, we get a matrix element that depends only on the difference in the indices of the initial row and column, i.e. all the elements of the resulting matrix will be equal on the corresponding diagonals.

The next step is to return to the initial form of the function, namely to the sum:

$$\vartheta(J_r(\lambda_1), J_r(\lambda_2)) = \begin{pmatrix} \vartheta(\lambda_1,\lambda_2) & \ldots & \sum_{j=0}^{m-l} \frac{1}{j!(m-l-j)!} \frac{\delta^{m-l}\vartheta(z,\tau)}{\delta z^{m-l-j}\delta\tau^j}\Bigg|_{\substack{z=\lambda_1 \\ \tau=\lambda_2}} \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \vartheta(\lambda_1,\lambda_2) \end{pmatrix};$$

$$\vartheta(Z,T) \underset{\text{commuting}}{\overset{\text{for } Z,T}{=}} U\vartheta(J_1,J_2)U^{-1}$$

3.2 The period of the Jacobi theta function

The scalar Jacobi theta function is periodic with a period of 1 in z: $\vartheta(z \pm 1, \tau) = \vartheta(z,\tau)$, and by completing the square, τ — quasiperiodic in z:

$$\vartheta(z \pm \tau, \tau) = e^{-\pi i\tau \mp 2\pi iz}\vartheta(z,\tau).$$

In the case of matrix variables, the 1-periodicity of the first variable is transformed into the periodicity of the unit matrix I:

$$\vartheta(J_r(\lambda_1) + I, J_r(\lambda_2)) = J_{r_1}(\lambda_1) + I = J_{r_1}(\lambda_1 + 1) = \vartheta(J_r(\lambda_1 + 1), J_r(\lambda_2)) =$$

$$= \begin{pmatrix} \vartheta(\lambda_1+1,\lambda_2) & \ldots & \sum_{j=0}^{m-l} \frac{1}{j!(m-l-j)!} \frac{\delta^{m-l}\vartheta(z,\tau)}{\delta z^{m-l-j}\delta\tau^j}\Bigg|_{\substack{z=\lambda_1+1 \\ \tau=\lambda_2}} \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \vartheta(\lambda_1+1,\lambda_2) \end{pmatrix} =$$

$$= \begin{pmatrix} \vartheta(\lambda_1,\lambda_2) & \dots & \left.\sum_{j=0}^{m-l} \frac{1}{j!(m-l-j)!} \frac{\delta^{m-l}\vartheta(z,\tau)}{\delta z^{m-l-j}\delta\tau^{j}}\right|_{\substack{z=\lambda_1 \\ \tau=\lambda_2}} \\ \vdots & \ddots & \vdots \\ 0 & \dots & \vartheta(\lambda_1,\lambda_2) \end{pmatrix} = \vartheta(J_r(\lambda_1), J_r(\lambda_2)).$$

Then for two arbitrary commuting matrices Z, T :

$$\vartheta(Z+I,T) = \sum_{n=-\infty}^{\infty} e^{2\pi in(Z+1)} e^{2\pi in^2 T} = \sum_{n=-\infty}^{\infty} e^{2\pi inZ} e^{2\pi inI} e^{2\pi in^2 T} =$$

$$= \sum_{n=-\infty}^{\infty} Ue^{2\pi inJ_1}U^{-1}e^{2\pi in\,I}Ue^{2\pi in^2 J_2}\mathrm{U}^{-1} = \sum_{n=-\infty}^{\infty} Ue^{2\pi in\,J_1}e^{2\pi in\,I}e^{2\pi in^2 J_2}\mathrm{U}^{-1} =$$

$$= \sum_{n=-\infty}^{\infty} Ue^{2\pi in\,(J_1+I)}e^{2\pi in^2 J_2}\mathrm{U}^{-1} = U\vartheta(J_r(\lambda_1)+I, J_r(\lambda_2))U^{-1} =$$

$$= U\vartheta(J_1,J_2)U^{-1} = \vartheta(Z,T).$$

4 COMPARISON OF THE OBTAINED RESULTS WITH EXISTING METHODS OF WORKING WITH MATRIX-VARIATE FUNCTIONS

4.1 Comparison of obtaining the matrix Gamma function using the Lanczos approximation method and the obtained method

Computing the matrix Gamma function by the Lanczos method [3] is performed on the basis of the following formula:

$$\Gamma(A) = \sqrt{2\pi}\left(A+\left(\alpha+\frac{1}{2}\right)I\right)^{A+\frac{1}{2}I} e^{-\left(A+\left(\alpha+\frac{1}{2}\right)I\right)} \times$$

$$\times\left(c_0(\alpha)I + \sum_{k=1}^{m} c_k(\alpha)(A+(k-1)I)^{-1} + e_{\alpha,m}(A)\right),$$

where $c_k(\alpha)$ are the Lanczos coefficients that depend on the parameter α .

Typically, pre-logarithmization is used to optimize calculations and avoid overflow problems during calculations:

$$\ln(\Gamma(A)) = \frac{1}{2}\ln(2\pi) + \left(A+\frac{1}{2}I\right)\ln\left(A+\left(\alpha+\frac{1}{2}\right)I\right) - \left(A+\left(\alpha+\frac{1}{2}\right)I\right) +$$

$$+\ln\left(c_0 cI + \sum_{k=1}^{m} c_k\alpha(A+(k-1)I)^{-1} + e_{\alpha,m}(A)\right).$$

It is also important to note that the set of coefficients $c_k(\alpha)$ is found empirically [3] and for the example for the pair $\alpha = 9,\ m = 10$ the following values are used:

k	$c_k(9)$
0	1.000000000000000174663
1	5716.400188274341379136
2	− 14815.30426768413909044
3	14291.49277657478554025
4	− 6348.160217641458813289
5	1301.608286058321874105
6	− 108.1767053514369634679
7	2.605696505611755827729
8	$-0.7423452510201416151\times10^{-2}$
9	$0.538413643250956406\times10^{-2}$
10	$-0.4023533141268236372\times10^{-2}$

Now let's compare the actual algorithms for finding matrices by these two methods:

Algorithm for finding the function using the Lanczos method	**Algorithm for finding the function using the Jordan form**
1. Set $\alpha = 9$; $m = 10$; $S = c_0 I + c_1 A^{-1}$;	1. Eigenvalues λ_i of matrix A;
2. for $k = 2:10$	2. Eigen and adjoint vectors x_i;
3. $S = S + c_k (A + (k-1)I)^{-1}$;	3. Jordan form J;
4. end	4. $\Gamma(J)$ and transitional matrix U based on x_i;
5. $L = \frac{1}{2}\ln(2\pi)I + \left(A - \frac{1}{2}I\right)\ln\left(A + \frac{17}{2}I\right) - -\left(A + \frac{17}{2}I\right) + \ln(S)$;	5. $\Gamma(A) = U\Gamma(j)U^{-1}$.
6. $\Gamma(A) \approx e^L$.	–

So, as we can see, the proposed algorithm is much more convenient for actually finding the values of the matrix Gamma function $\Gamma(A)$ in comparison with some existing numerical methods. It is also in addition to the above that our method has the advantage of being able to use the obtained function and its properties in further research. Similar results were obtained for Spouge's approximation method [3], since both of them have similar algorithms.

4.2 Research using the Schur decomposition

The Schur decomposition method [4] is based on the decomposition of the input matrix and its representation through unitary and upper triangular:

$$\forall A, B \in \mathbb{C}(n \times n) : A * B = B * A,\ \exists U,\ R_1, R_2 : A = QR_1Q^*;\ B = QR_2Q^*.$$

Thanks to this, in their work L. Jódar, J. CCortés [5] for two commuting matrices proved several properties of the matrix-variate Beta function, namely the symmetry of the variables and the connection with the matrix Gamma function:

$$B(P,Q)=\Gamma(P)\Gamma(Q)\Gamma-1(P+Q).$$

It should be noted that the last property was proved only for diagonalizable, commuting matrces P and Q. Compared to the obtained results, we coan sii that the main advantage of using the Jordan canonical form is the presence of an explicit form of the resulting matrix. This, in turn, gives us the following advantages compared to the Shur Schedule:

- Ability to derive properties associated with certain partial cases and specific function values;
- From the point of view of computational complexity, although historically the calculation of the Jordan canonical form was usually considered a very difficult task, the properties of the matrix function from the Jordan matrix allow us to bypass this step, and so the need remains only to find the eigenvalues and the corresponding vectors to form a basis. Then, comparing to the Schur decomposition, which has a computational complexity of $O(n^3)$, our method will have an approximate complexity of $O(n^{\omega})$, $2<\omega<2.376$.

4.3 The zonal polynomials method

The method of zonal polynomials [6] is one of the methods for studying such functions using integrals and the difference in approach will be illustrated on its example.

In this method, the studied function differs from others, namely, it has the following form:

$$B_m(a,b)=\int_0^{I_m}\det(X)^{a-\frac{(m+1)}{2}}\det(I_m-X)^{b-\frac{(m+1)}{2}}dX.$$

Additional results and generalizations of this function were found using zonal polynomials and evaluating the resulting integral for them. The main use case of it and its generalized forms is the matrix beta distribution:

For $U\sim B_p^I(a,b)$, the distribution density of the positive definite square matrix U:

$$f(U)=\frac{1}{B_p(a,b)}\det(U)^{a-\frac{p+1}{2}}\det(I_p-U)^{b-\frac{p+1}{2}}.$$

As we can see, this function and similar functions of this type contain only matrix determinants and, in some cases, trace. This means that these functions are limited to uses only in problems in which the input signal has a matrix form, and the output signal is already scalar. This has a number of disadvantages in solving some statistical problems in which it is important to leave connections between certain vectors or blocks of vectors, like the problems that were mentioned in the introductory section.

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INFORMATION ON THE ARTICLE

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КЛАСИЧНІ СПЕЦІАЛЬНІ ФУНКЦІЇ З МАТРИЧНИМИ ЗМІННИМИ / Д.О. Шутяк, Г.Б. Подколзін, В.Г. Бондаренко, Ю.А. Чаповський

Анотація. Розглянуто декілька найбільш часто використовуваних спеціальних функцій та їх ключові властивості, а також запропоновано аналітичний підхід до побудови їх аналогів із метричними змінними. Щоб досягти цього, ми уникали використання будь-яких алгоритмів чисельного наближення та натомість покладались на властивості матриць, матричної експоненти та Жорданову нормальну форму для представлення матриць. Ми зосередились на таких функціях: гамма-функція як приклад функції однієї змінної з великою кількістю властивостей і застосувань; бета-функція, щоб підкреслити подібності та відмінності від додавання другої змінної до функції матричної змінної; тета-функція Якобі. Побудовано явні представлення функцій і доведено декілька ключових властивостей для цих функцій; висвітлено та порівняно інші підходи, які використовувалися в минулому для вирішення цих задач.

Ключові слова: матриця, спеціальна функція, гамма-функція, бета-функція, тета-функції Якобі, Жорданова нормальна форма.