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IDENTIFICATION OF NONLINEAR SYSTEMS WITH PERIODIC EXTERNAL ACTIONS (PART III)

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Abstract. The article considers the problem of identifying a mathematical model in the form of a system of ordinary differential equations. The identified system can have constant and periodic coefficients. The source of information for solving the problem is time series of observed variables. The article studies the effect of uniformly distributed noise on the identification result. To solve the problem, the algorithm proposed by the author in previous works was used. It is shown that the method has different sensitivity to noise depending on which of the observed variables is contaminated with noise. The implementation of the method is illustrated by numerical examples of identifying nonlinear differential equations with polynomial right-hand sides.

Keywords: identification, ordinary differential equation, periodic coefficient, constant coefficient, uniformly distributed noise.

INTRODUCTION

When developing mathematical methods for studying various physical systems, it is necessary to evaluate reliability of their results if applied to systems in real world. One of the common tasks in applied mathematics is the problem of identifying a mathematical model of a certain process. The initial data for solving this problem can be the observed variables of the process. If we study real systems, the results of measurements of the observed variables may contain noise [1–3]. This circumstance can complicate the identification of the model.

BACKGROUND AND TASK FOR RESEARCH

We follow the results obtained in [4; 5]. There, the problem of identifying a system of n ordinary differential equations with constant and periodic coefficients $c_{ij}(t)$ ($i=1,...,n$; $j=1,...,m$) was considered. The initial data for identification was time series of observed variables $x_i(t)$, $t \in [0;t_e], t_e > 0$. To solve the problem, the theorem proved in [4] was used. According to this theorem, simple relationships that are used to identify equations with constant coefficients can be used to identify differential equations with periodic coefficients. For this purpose, the calculations must use the values of the functions $x_i(t)$ at moments of time t_j, separated from each other by the value qT, $q \in 1,2,3,...$, where T is the period of the periodic coefficients. In other words, this time moments obey the relations

$$t_1 = t_0 + \tau,\ t_2 = t_0 + 2\tau,\ ...,\ t_m = t_0 + m\tau;\ \ t_0 \ge 0,\ \tau > 0,\ t_m \le t_e, \qquad (1)$$

where $\tau = qT$.

Examples of the method application were demonstrated in [4; 5]. In this study, we try to apply the proposed algorithm to identify equations by observed variables with noise. As in the mentioned articles, we use as an example a system of the form

$$\begin{cases} \dot{x}_1 = -x_2 - x_3, \\ \dot{x}_2 = x_1 - dx_2, \\ \dot{x}_3 = c_{30}(t) + c_{33}(t)x_3 + c_{36}(t)x_1x_3, \end{cases} \tag{2}$$

obtained on the basis of the Rössler system [6]. The parameters of the third equation of the system (2) are as following:

$$c_{30}(t) = 0.5 + 0.4\sin\left(\frac{2\pi t}{T_0}\right),\ c_{33}(t) = -20,\ c_{36}(t) = 5,\ T_0 = 2\,s.$$

The generalized structure for the purpose of identifying the desired equation has the form

$$\dot{x}_3 = c_{30}(t) + c_{31}(t)x_1 + c_{32}(t)x_2 + c_{33}(t)x_3 + c_{34}(t)x_1^2 + c_{35}(t)x_1x_2 +$$

$$+ c_{36}(t)x_1x_3 + c_{37}(t)x_2^2 + c_{38}(t)x_2x_3 + c_{39}(t)x_3^2. \tag{3}$$

Next, we will consider how adding noise to different observables affects the identification result.

IDENTIFICATION OF AN EQUATION WITH A VARIABLE $x_1(t)$ AFFECTED BY NOISE

For the study, we add noise with a uniform distribution to the observable $x_1(t)$ **.** The noise value is $\Delta u_1 = 0.01 \cdot \Delta x_1$, $\Delta x_1 = |x_{1\max} - x_{1\min}|$, $x_{1\max}$ and $x_{1\min}$ are, respectively, the maximum and the minimum of the observable $x_1(t)$ over the studied interval of 100 s. Fig. 1 shows a fragment of the time series $x_1(t)$ with added noise.

The first stage of model identification with this algorithm is to find those values of τ (see (1)) that can be equal to or multiples of the expected T. The values obtained by applying the algorithm are presented in Table 1. The table shows in bold the τ values that are repeated or multiples of other values. This means that they may be the sought-for values of T or multiples of this value. For example, the values 1.04, 3.01, 4.88, 5.30, 5.67, 10.96 are repeated. The set: 2.65, 5.30, 10.60 is also highlighted in bold because the second and the third values of it are multiples of the first one. Similarly, the value 6.00 is a multiple of 3.00. The values 2.20

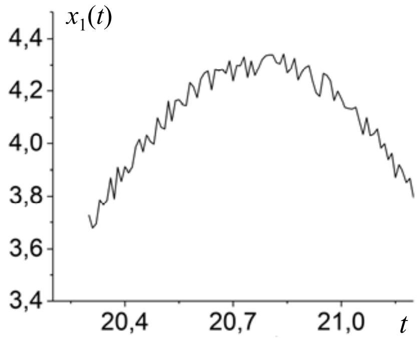

Fig. 1. Time series of $x_1(t)$ contaminated with noise

and 7.70 are multiples of 1.10, which is not in the table, but which may potentially be the sought-for period. For the same reason, 6.00, 8.00, 10.00 are highlighted, which are multiples of 2.00, which is not in the table.

Table 1. The result of applying the algorithm for observable $x_1(t)$ with noise

№	The τ values calculated for the coefficients of the third equation of system (2)									
	$c_{30}(t)$	$c_{31}(t)$	$c_{32}(t)$	$c_{33}(t)$	$c_{34}(t)$	$c_{35}(t)$	$c_{36}(t)$	$c_{37}(t)$	$c_{38}(t)$	$c_{39}(t)$
1	9.25	5.24	5.18	4.08	5.95	7.26	1.84	**6.00**	1.72	2.63
2	0.88	1.27	3.34	1.29	5.31	5.86	**2.65**	6.03	3.66	0.86
3	**7.70**	**10.96**	**3.00**	8.18	8.98	**6.00**	5.18	6.64	**3.01**	**5.67**
4	**1.04**	3.31	9.91	2.98	5.91	3.20	5.36	0.81	5.85	2.58
5	8.23	2.30	10.94	10.95	3.33	7.72	**5.67**	7.07	8.23	7.45
6	**3.01**	4.36	5.86	**3.01**	4.74	**5.30**	5.39	7.79	**10.60**	1.91
7	9.17	**4.88**	5.91	5.32	**10.96**	4.97	9.07	8.92	9.06	1.08
8	4.81	**10.00**	1.85	3.49	9.80	5.44	**3.01**	7.99	**6.00**	7.44
9	**4.88**	**2.20**	4.79	7.58	**8.00**	5.98	5.91	1.53	10.66	9.24
10	2.98	4.43	7.99	**5.30**	1.48	7.59	1.47	9.50	4.02	**1.04**

We have to, based on the data in Table 1, reject the excess coefficients of the desired equation and estimate the type and values of the remaining coefficients. For this, we use the second part of the algorithm. Namely, for each value selected in Table 1, we solve a system of the form

$$\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}, \tag{4}$$

where $\mathbf{C}$ is the vector of the required coefficients of the third equation of system (2), and $\mathbf{B}$ is the vector of values $\dot{x}_i(t_k)$, $k = 0,...,m$, $\mathbf{A}$ is the matrix of function $f_j(\mathbf{x}(t_k))$ values, $j = 0,...,m$, $\mathbf{x} = \{x_1,...,x_n\}$. In this study we consider as functions $f_j(\mathbf{x}(t_k))$ the products of x_i in each monomial of the right-hand side of (3). We set some interval of change t_0 from (1) and obtain the calculated values of the coefficient on this interval. The resulting time series of coefficients allow us to estimate the type of coefficient and the possible value of T.

To begin with, let us try to identify zero coefficients in the analyzed equation, provided that the selected value of τ from Table 1 can be the desired period T. For example, Fig. 2 shows the time series of the calculated coefficient $c^c_{34}(t)$ for $\tau = 1.04\ s$ on an interval of 4 s.

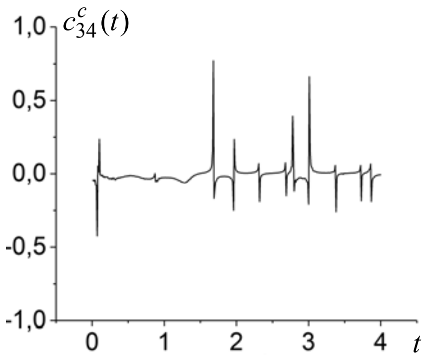

Fig. 2. Time series of calculated coefficient $c^c_{34}(t)$

As can be seen, this coefficient has values close to zero on this segment. At the same time, its greatest deviation from zero, including at singular points, is $|c^c_{34}(t)| < 1$. We will assume that if these conditions are met, this coefficient is a candidate for zeroing. Such a criterion was applied to all coefficients for all selected τ from Table 1. The results of the analysis are presented in Table 2, where the coefficients that may be zero in the desired equation are marked with a "+" sign. Note that this table has been supplemented with the values $\tau = 1.10\ s$ and $\tau = 2.00\ s$, which, as explained above, may also be values of the period T .

T a b l e 2 . Results of the analysis of time series obtained for the selected values of τ from Table 1

№	**Possible values of *T* or its multiples**															
	1.04	1.10	2.20	7.70	2.00	6.00	8.00	10.00	3.00	3.01	2.65	5.30	10.60	4.88	5.67	10.96
c_{34}	+				+											
c_{35}					+	+			+				+	+		
c_{37}	+				+	+		+	+				+	+		+

According to Table 2, the most likely candidates for zeroing are coefficients c_{35} and c_{37} . Taking $c_{35} = c_{37} = 0$ and performing an analysis similar to the previous one, we obtain Table 3.

T a b l e 3 . The same as in Table 2 with $c_{35} = c_{37} = 0$

№	**Possible values of *T* or its multiples**									
	1.04	2.00	4.00	6.00	8.00	10.00	3.00	10.60	4.88	10.96
c_{31}					+					
c_{32}		+		+	+					
c_{34}		+	+	+	+	+				+

Note that Table 3 does not include the τ values from Table 2, for which (according to Table 2) it is not possible to determine the coefficients that may be subject to zeroing. It should be noted that the value $\tau = 4.00s$ is added to the table because it is a multiple of $\tau = 2.00s$. One can also pay attention to the graph $c_{30}(t)$ obtained, for example, at $\tau = 4.00s$ (see Fig. 3). We can already assume from it that $T = 2.00s$.

Based on Table 3, the next step should be to zero out the coefficients included in it. As a result, time series of the remaining coefficients $c_{30}^c(t)$, $c_{33}^c(t)$, $c_{36}^c(t)$, $c_{38}^c(t)$, $c_{39}^c(t)$ were obtained.

For further analysis, let us consider graphs $c_{38}^c(t)$ and $c_{33}^c(t)$ shown in Fig. 4. Despite the large number of points in which the calculated value of the coefficient $c_{38}^c(t)$ deviates significantly from zero, we can assume that $c_{38} = 0$. Graph $c_{39}^c(t)$ looks similar, which also allows us

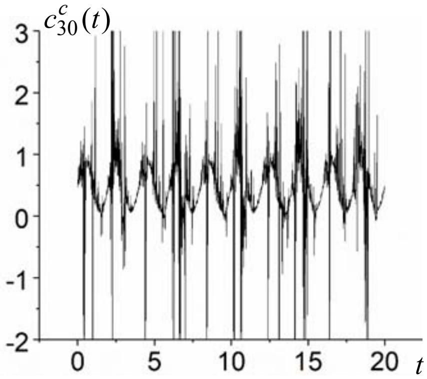

Fig. 3. Time series of calculated coefficient $c_{30}^c(t)$

to exclude it from the desired equation. On the contrary, most points of graph $c_{33}^{c}(t)$ clearly have values different from zero. Graph $c_{36}^{c}(t)$ looks similar to $c_{33}^{c}(t)$. Therefore, at the next step, it is advisable to zero out the coefficients c_{38} and c_{39}. As a result, this method gives us the structure of equation analogous to the apriori equation (2). Time series of calculated coefficients are shown in Fig. 5.

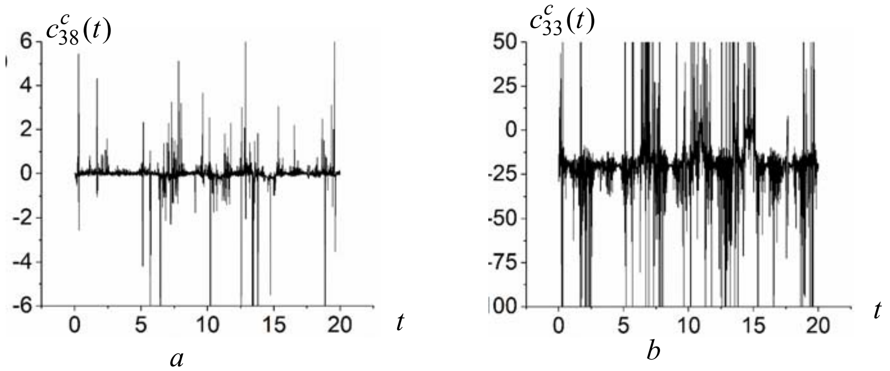

Fig. 4. Time series of calculated coefficients $c_{38}^{c}(t)$ and $c_{33}^{c}(t)$

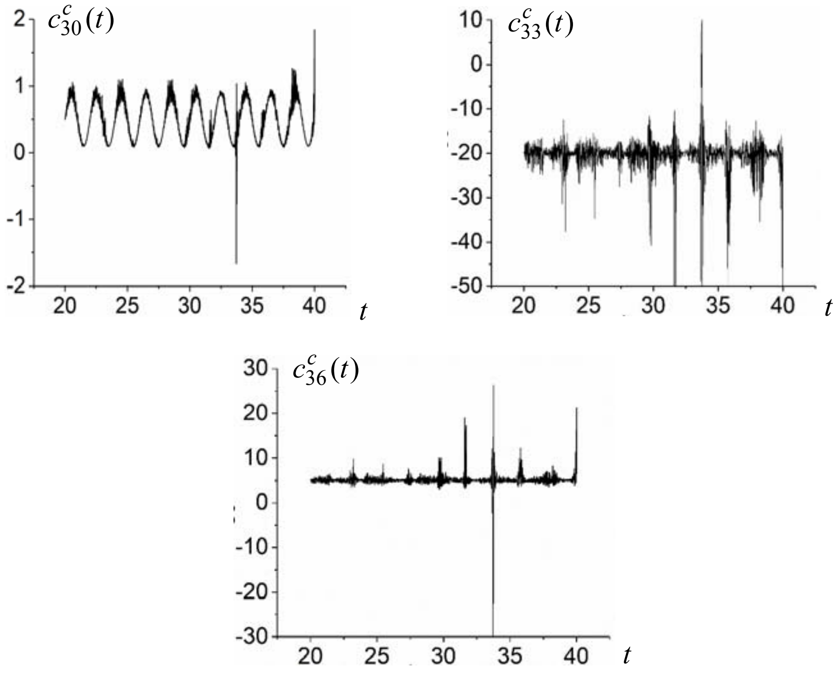

Fig. 5. Time series of calculated coefficients of the identified equation with the desired structures

IDENTIFICATION OF AN EQUATION WITH A VARIABLE $x_2(t)$ AFFECTED BY NOISE

A study similar to that performed in the previous section was also performed for a variable with the same noise level $\Delta u_2 = 0.01 \cdot \Delta x_2$. The result is shown in Table 4.

Table 4. The result of applying the algorithm for observable $x_2(t)$ with noise 1%

№	The τ values calculated for the coefficients of the third equation of system (2)									
	$c_{30}(t)$	$c_{31}(t)$	$c_{32}(t)$	$c_{33}(t)$	$c_{34}(t)$	$c_{35}(t)$	$c_{36}(t)$	$c_{37}(t)$	$c_{38}(t)$	$c_{39}(t)$
1	2.98	**4.00**	**10.00**	**4.00**	**4.00**	**8.00**	**4.00**	**10.00**	**2.00**	**4.00**
2	2.76	**8.00**	**2.00**	**8.00**	**8.00**	**10.00**	**8.00**	**2.00**	**10.00**	**10.00**
3	1.88	**10.00**	**8.00**	**10.00**	**10.00**	**2.00**	**10.00**	**4.00**	**8.00**	**8.00**
4	3.34	**2.00**	7.63	**2.00**	**2.00**	**4.00**	**2.00**	**6.00**	**4.00**	0.86
5	8.71	1.27	**4.00**	10.95	9.80	5.91	1.88	**8.00**	10.67	9.24

Based on the corollary of Theorem [4], we can conclude that the equation has a single variable coefficient $c_{30}(t)$, and the other coefficients are constant. That is, in this case, the identification occurs in the same way as for equations without noise, see [4, 5]. Moreover, a similar result was obtained by increasing the noise level added to the observable $x_2(t)$. Fig. 6 shows this variable with noise $\Delta u_2 = 0.2 \cdot \Delta x_2$, and Table 5 demonstrates the result of applying the algorithm.

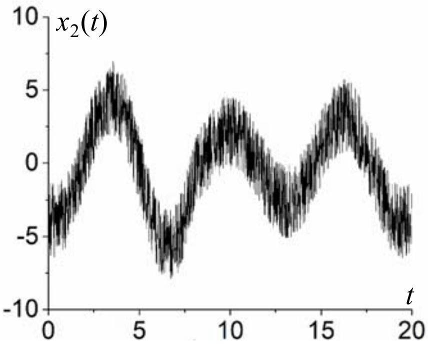

Fig. 6. Time series of $x_2(t)$ contaminated with noise 20%

Table 5. The result of applying the algorithm for observable $x_2(t)$ with noise 20%

№	The τ values calculated for the coefficients of the third equation of system (2)									
	$c_{30}(t)$	$c_{31}(t)$	$c_{32}(t)$	$c_{33}(t)$	$c_{34}(t)$	$c_{35}(t)$	$c_{36}(t)$	$c_{37}(t)$	$c_{38}(t)$	$c_{39}(t)$
1	10.75	**8.00**	**6.00**	**8.00**	**8.00**	**4.00**	10.35	**6.00**	**6.00**	**8.00**
2	1.20	**10.00**	**4.00**	1.46	**10.00**	**6.00**	**8.00**	**10.00**	**10.00**	**10.00**
3	1.05	**2.00**	**10.00**	**10.00**	**2.00**	**10.00**	**10.00**	**8.00**	**4.00**	6.73
4	6.89	5.29	**8.00**	0.72	**4.00**	**8.00**	0.95	**2.00**	**8.00**	**2.00**
5	2.60	1.86	**2.00**	8.75	**6.00**	**2.00**	**4.00**	**4.00**	**2.00**	**4.00**

Based on the comparison of these two tables, it can be concluded that the algorithm has low sensitivity to noise in this case. This can also be illustrated by Table 6, which presents the calculated values of the constant coefficients at 20% noise for two different t_0. It is clear from the table that in the structure (3) all terms that include the variable x_2 are subject to zeroing. Therefore, in the subsequent steps of identification, only the observables x_1 and x_3 will be used, which in this case are noise-free. This simplifies the task.

Table 6. The calculated values of the constant coefficients at 20% noise for two different t_0

t_0	Coefficients								
	c_{31}	c_{32}	c_{33}	c_{34}	c_{35}	c_{36}	c_{37}	c_{38}	c_{39}
$t_{01} = 0.15s$	-1.307 10^{-5}	-7.991 10^{-6}	-20.008	-1.288 10^{-6}	2.425 10^{-6}	5.002	1.367 10^{-7}	3.321 10^{-4}	4.862 10^{-4}
$t_{02} = 0.4s$	-3.51 10^{-6}	2.514 10^{-6}	-20.026	8.848 10^{-7}	-3.002 10^{-7}	5.006	3.638 10^{-7}	6.584 10^{-4}	1.585 10^{-3}

IDENTIFICATION OF AN EQUATION WITH A VARIABLE $x_3(t)$ AFFECTED BY NOISE

Unlike the previous case, the variable x_3 is present in the desired equation, both in the right-hand side and in the left-hand side in the form of a time derivative. A fragment of the noisy series $x_3(t)$ with $\Delta u_3 = 0.01 \cdot \Delta x_3$ is shown in Fig. 7, *a*.

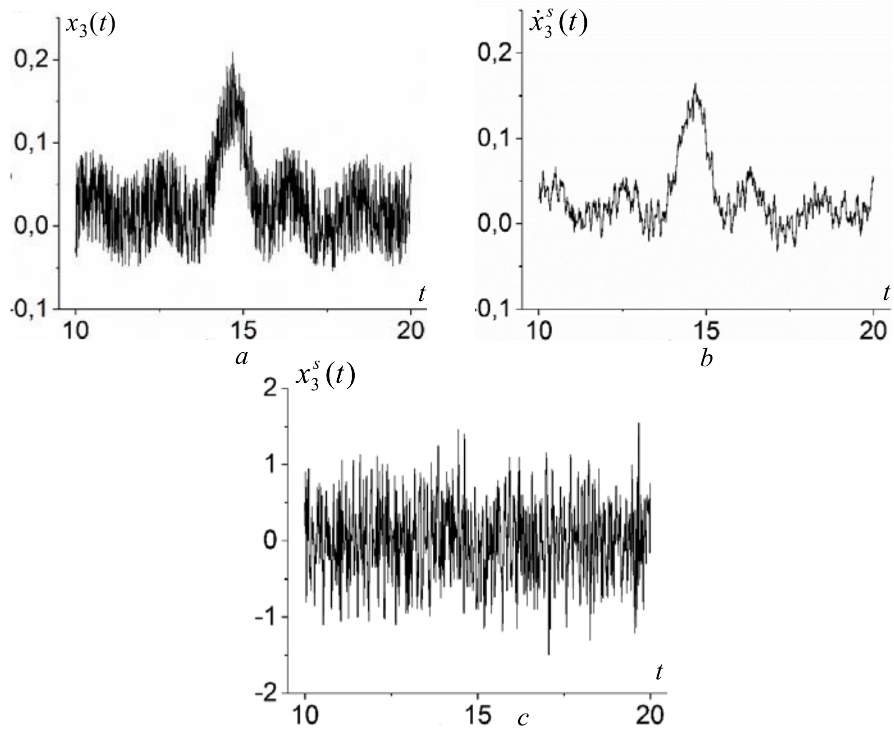

Fig. 7. Time series of $x_3(t)$: *a* — fragment of time series $x_3(t)$ contaminated with noise 1%; *b* — the same fragment after smoothing; *c* — time derivative of graph (*b*)

In order to reduce noise, smoothing was performed using the moving average method according to the formula

$$x_{3\nu}^s = \frac{x_{3\nu-3} + x_{3\nu-2} + x_{3\nu-1} + x_{3\nu} + x_{3\nu+1} + x_{3\nu+2} + x_{3\nu+3}}{7}, \tag{5}$$

where $x_{3\nu}^s$ is the value of the function $x_3(t)$ at the point with the number ν after smoothing by formula (5). A fragment of the function $x_3^s(t)$ is shown in Fig. 7, *b*. To form the vector **B** of the left sides in the system (4), it is necessary to perform numerical differentiation of the function $x_3^s(t)$. For this, the formula

$$\dot{x}_{3\nu}^s = \frac{x_{3\nu+1}^s - x_{3\nu-1}^s}{2\Delta t}$$

was used, where Δt is a step of time series $x_3(t)$ representation. The time series $\dot{x}_{3\nu}^s(t)$ is shown in Fig. 7, *c*, which demonstrates that noise is significantly amplified when numerical differentiation is computed.

Unfortunately, the application of the algorithm did not allow us to obtain an adequate result. Obviously, the reason for this is insufficient smoothing of the time series noise and the appearance of significant computational noise during numerical differentiation.

DISCUSSION AND CONCLUSIONS

From the previous sections it is clear that the most difficult identification is when the variables $x_1(t)$ and $x_3(t)$, which are included in the desired equation, are contaminated with noise. The results obtained can be explained using Cramer's rule. When determining the coefficients of equation (3), it will have the form:

$$c_{3j} = \frac{\det(\mathbf{A}_j)}{\det(\mathbf{A})}, \; j = 0,\ldots,m,$$

where $\det(\mathbf{A})$ is the main determinant of the system of linear algebraic equations (4), formed taking into account conditions (1), $\det(\mathbf{A}_j)$ is the determinant obtained by replacing the *j*-th column of the determinant $\det(\mathbf{A})$ with the vector $\mathbf{B}$ from (4).

Let the variable with noise be $x_1(t)$. Then, to find, for example, the coefficient c_{31}, we create matrix $\mathbf{A}_1$ by replacing column 1 with $\mathbf{B}$ in matrix $\mathbf{A}$, which in this case consists of the time derivatives of the values of $x_3(t)$. That is, we replace the column of values of noisy $x_1(t)$ with the column $\mathbf{B}$ without noise. Here we assume that the differentiation of the variable $x_3(t)$ without noise is performed correctly, without significant errors. Therefore, such a replacement, at a minimum, should not increase the error in calculating c_{31}.

On the contrary, if the observed variable with noise is $x_3(t)$, then when differentiating it numerically, computational noise will appear, see Fig. 7, *c*. Then, when calculating, for example, the coefficient c_{33}, we replace the column number 3 in $\mathbf{A}$ with vector $\mathbf{B}$. The column number 3 of $\mathbf{A}$ initially contains the variable $x_3(t)$, which contains noise. However, comparing Fig. 7, *b* and 7, *c*, we see that the noise in the column $\mathbf{B}$ is much greater and can of course be a source of significant errors.

Taking into account the above, for successful identification of systems from time series of observations with noise, sometimes it is necessary to use various noise filtering methods [7–12] that are more effective than (5). The use of more effective, although more complex, methods of numerical differentiation [13–15] is also justified.

REFERENCES

1. P.R. Deboeck, S.M. Boker, "Modeling noisy data with differential equations using observed and expected matrices," *Psychometrika*, vol. 75, no. 3, pp. 420–437, Sept. 2010. doi: 10.1007/S11336-010-9168-2
2. H. Liang, H. Miao, and H. Wu, "Estimation of constant and time-varying dynamic parameters of HIV infection in a nonlinear differential equation model," *Ann. Appl. Stat.*, no. 4(1), pp. 460–483, Mar. 1, 2010. doi: 10.1214/09-AOAS290
3. J.O. Ramsay, G. Hooker, D. Campbell, and J. Cao, "Parameter estimation for differential equations: a generalized smoothing approach," *J. Roy. Statistic. Soc., Ser. B*, vol. 69, no. 5, pp. 741–796, Nov. 2007. doi: 10.1111/j.1467-9868.2007.00610.x

4. V. Gorodetskyi, “Identification of nonlinear systems with periodic external actions (Part 1),” *System Research & Information Technologies*, no. 3, pp. 93–106, 2024. doi: 10.20535/SRIT.2308-8893.2024.3.06
5. V. Gorodetskyi, “Identification of nonlinear systems with periodic external actions (Part 2),” *System Research & Information Technologies*, no. 4, pp. 66–76, 2024. doi: 10.20535/SRIT.2308-8893.2024.4.05
6. O.E. Rössler, “An equation for continuous chaos,” *Phys. Lett. A.*, vol. 57, no. 5, pp. 397–398, 1976. doi: 10.1016/0375-9601(76)90101-8
7. M. Mafi, H. Martin, M. Cabrerizo, J. Andrian, A. Barreto, and M. Adjouadi, “A comprehensive survey on impulse and Gaussian denoising filters for digital images,” *Sign. Proc.*, vol. 157, pp. 236–260, Apr. 2019. doi: 10.1016/j.sigpro.2018.12.006
8. L. Tan, J. Jiang, “Finite Impulse Response Filter Design,” in *Digital Signal Processing (Third Edition): Fundamentals and Applications*, 2019, pp. 229–313.
9. J.P. do Vale Madeiro, P.C. Cortez, J.M. da Silva Monteiro Filho, and P.R.F. Rodrigues, “Techniques for Noise Suppression for ECG Signal Processing,” in *Developments and Applications for ECG Signal Processing. Modeling: Segmentation, and Pattern Recognition*, pp. 53–87, 2019. doi: 10.1016/B978-0-12-814035-2.00009-8
10. S.V. Vaseghi, *Advanced Digital Signal Processing and Noise Reduction. (Fourth Edition)*. London, UK: Wiley and Sons, 2008.
11. Y. Chen, S. Fomel, “Random noise attenuation using local signal-and-noise orthogonalization,” *Geophys.*, vol. 80, no. 6, pp. WD1–WD9, Nov.-Dec., 2015. doi: 10.1190/GEO2014-0227.1
12. T. Kelemenová, O. Benedik, and I. Koláriková, “Signal noise reduction and filtering,” *Acta Mechatronica*, vol. 5, no. 2, pp. 29–34, Jun. 2020. doi: 10.22306/am.v5i2.65
13. I. Knowles, R.J. Renka, “Methods for numerical differentiation of noisy data,” *Electr. J. Dif. Eq.,* Conference 21, pp. 235–246, 2014, presented at Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems, 2012.
14. R. Chartrand, “Numerical differentiation of noisy, nonsmooth data,” *Intern. Schol. Res. Netw. Appl. Math.,* vol. 2011, Art. no. 164564. doi: 10.5402/2011/164564
15. K. Ahnert, M. Abel, “Numerical differentiation of experimental data: local versus global methods,” *Comput. Phys. Commun.* vol. 177, no. 10, pp. 764–774, Nov. 2007. doi: 10.1016/j.cpc.2007.03.009

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INFORMATION ON THE ARTICLE

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ІДЕНТИФІКАЦІЯ НЕЛІНІЙНИХ СИСТЕМ З ПЕРІОДИЧНИМИ ЗОВНІШНІМИ ДІЯМИ (Частина ІІІ) / В.Г. Городецький

Анотація. Розглянуто проблему ідентифікації математичної моделі у вигляді системи звичайних диференціальних рівнянь. Ідентифікована система може мати сталі та періодичні коефіцієнти. Джерелом інформації для розв’язання поставленої задачі є часові ряди спостережуваних змінних. Досліджено вплив шуму з рівномірним розподілом на результат ідентифікації. У ході дослідження використовувався алгоритм, запропонований автором у попередніх працях. Розглянуто особливості застосування цього алгоритму для даної задачі. Показано, що метод має різну чутливість до шуму залежно від того, яка із спостережуваних змінних забруднена шумом. Реалізацію методу проілюстровано чисельними прикладами ідентифікації нелінійних диференціальних рівнянь із поліноміальними правими частинами.

Ключові слова: ідентифікація, звичайне диференціальне рівняння, періодичний коефіцієнт, сталий коефіцієнт, шум із рівномірним розподілом.