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MATRIX-GRAPHIC SIMULATION OF SOCIAL NETWORK: ERGODIC PROPERTIES

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Abstract. We propose mathematical tools for a social network simulation in order to obtain some sufficient conditions of the network's ergodicity, that is, the existence of a steady state as $t \to +\infty$. The proposed model is linear; the elements of the network form a two-dimensional array (i.e., a matrix) $\Omega = \{1,2,\dots,n\} \times \{1,2,\dots,m\}$, where $A_{i,j}(t) \in [0,1]$ is the state of the element $(i,j) \in \Omega$, $t \geq 0$ is time. An impact operator T is a four-dimensional array; the element $T_{i,j,k,l} \geq 0$ denotes the impact of the element (i,j) on the element (k,l): $(TA)_{i,j} = \sum_{k=1}^{n} \sum_{l=1}^{m} T_{i,j,k,l} A_{k,l}$. The impact operator T is also presented as a directed graph G_T, whose vertices correspond to elements $(i,j) \in \Omega$: a directed edge (an arc) leads from the vertex $(k,l) \in \Omega$ to the vertex $(i,j) \in \Omega$ if and only if $T_{i,j,k,l} > 0$, and this edge is labelled by the number $T_{i,j,k,l}$. A bound $B \subset \Omega$ is introduced in such a way that $T_{i,j,k,l} = 0$ for $(k,l) \in \Omega$, $(i,j) \in B$. The state $A(t+1)$ at time $t+1$ is defined by the state $A(t)$ at the current time $t \geq 0$ via equation $A(t+1) = TA(t) + \Delta$, where matrix Δ of dimension $n \times m$ defines the states of bound elements $(i,j) \in B$; $\Delta_{i,j} = 0$ for internal elements $(i,j) \in \Omega \setminus B$. Some sufficient conditions for the network's ergodicity are given in the form of connectivity properties of the impact graph G_T. This graph must contain paths between all pairs of vertices and loops for all vertices. Suggested conditions provide the spectrum of T (with the possible exception of $\lambda = 1$) to be located inside the open unit disk; we prove that $\lambda = 1$ is an eigenvalue of T if and only if the bound $B \subset \Omega$ is isolated (no bound element impacts any internal one). These spectral properties of T provide that the steady state exists and can be found by the iterative procedure $A(t+1) = TA(t) + \Delta$ with the given $A(0)$; the iterative process converges linearly (geometrically).

Keywords: social system, simulation, ergodicity, eigenvalue, Jordan normal form.

INTRODUCTION

Social network analysis is currently one of the most important methods for scientific investigation in sociology, social psychology and other areas (see, e. g., [1; 2]). A social network is defined by the interaction of network elements, or, in other words, by impact of network elements on other ones.

Various toolkits can be used to simulate a social network. For example, in [3; 4] graph theory methods are used to visualize network elements interaction, in [1] matrix analysis gives a more convenient way to analyze network elements interaction.

Usually, a social network is not a static structure, i.e. the state of network elements changes over time. The social network' behaviour is currently being intensively investigated (see, e.g., [3; 5; 6]), and steady states are of a special interest (see, e.g., [3]).

The purpose of this work is to obtain some sufficient conditions for social network ergodicity (independence of a network's behaviour from initial conditions in extremely distant time), using matrix and graph methods of social network representation.

REPRESENTATION OF A SOCIAL NETWORK AND ITS DYNAMICS

Suppose that the social network (hereinafter referred to as the network) contains nm elements, arranged in n rows and m columns ($n, m \in \mathbb{N}$), i.e. position of each element is defined by a pair $(i, j) \in \Omega$, where $\Omega = \{1,2,\dots,n\} \times \{1,2,\dots,m\} = \{(i, j) : 1 \le i \le n,\ 1 \le j \le m\}$ is the set (area) of coordinates of network elements.

The current state of the element (i, j) is defined by a number $A_{i,j}(t) \in [0,1]$, which can be particularly treated as an attitude of the element (i, j) towards some problem arisen in the network (0 means completely negative attitude, 1 means completely positive one); hereinafter $t \in \mathbb{N}_0$ denotes discrete time ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). Therefore, the current state of the network can be represented as a matrix $A(t) \in M_{n\times m}[0,1]$ of dimension $n \times m$ with elements $A_{i,j}(t) \in [0,1]$ ($(i, j) \in \Omega$).

Let $B \subset \Omega$ be the bound of the coordinate area Ω. The states of boundary elements are described by the boundary condition matrix Δ of dimension $n \times m$, assuming $\Delta_{i,j} = 0$ for all $(i, j) \in \Omega \setminus B$. Elements $(i, j) \in \Omega \setminus B$ that do not belong to B are called internal. Hereinafter, assume that $B \ne \Omega$ (excluding a trivial case $B = \Omega$).

In order to simulate network dynamics, introduce a linear impact operator $T : M_{n\times m}[\mathbb{R}] \to M_{n\times m}[\mathbb{R}]$, where $M_{n\times m}[\mathbb{R}]$ denotes a linear space of $n\times m$ matrices with elements from $\mathbb{R}$. The operator T is considered as 4-dimensional $n\times m\times n\times m$ array with elements from $\mathbb{R}$, its action on a matrix $X \in M_{n\times m}[\mathbb{R}]$ is defined point-wise:

$$(TX)_{i,j} = \sum_{k=1}^{n} \sum_{l=1}^{m} T_{i,j,k,l} X_{k,l} \, , \tag{1}$$

the element $T_{i,j,k,l}$ ($(i, j, k, l) \in \{1,2,\dots,n\} \times \{1,2,\dots,m\} \times \{1,2,\dots,n\} \times \{1,2,\dots,m\}$) defines impact of the state of the network element (k,l) on the state of the network element (i, j). The following conditions are assumed for normalization reasons:

$$\forall (i, j, k, l) \in \{1,2,\dots,n\} \times \{1,2,\dots,m\} \times \{1,2,\dots,n\} \times \{1,2,\dots,m\} : T_{i,j,k,l} \ge 0 \, ; \tag{2}$$

$$\forall (i, j) \in \Omega \setminus B : \sum_{k=1}^{n} \sum_{l=1}^{m} T_{i,j,k,l} = 1 \, . \tag{3}$$

Since the states of elements on the bound B are defined by matrix Δ, assume that

$$\forall (i, j) \in B \ \forall (k, l) \in \Omega : T_{i,j,k,l} = 0 \, . \tag{4}$$

Given relation (1), this means that $(TX)_{i,j} = 0$ for all $(i, j) \in B$.

Assume that the network state $A(t+1)$ at time $t+1$ ($t \ge 0$) is defined by the network state $A(t)$ at time $t \ge 0$ according to the equation

$$A(t+1) = TA(t) + \Delta, \tag{5}$$

the network state $A(0)$ at the initial time $t = 0$ is assumed to be defined by the initial condition matrix $A(0)$ of dimension $n \times m$ with elements $(A(0))_{i,j} = A_{i,j}(0) \in [0,1]$ ($(i, j) \in \Omega$). Note that the summand $TA(t)$ in relation (5) defines the states of internal elements $(i, j) \in \Omega \setminus B$, the summand Δ defines the states of elements $(i, j) \in B$ on the bound B.

The correspondence of the initial condition $A(0)$ with the boundary condition Δ (on the bound B) requires assumption

$$(A(0))_{i,j} = \Delta_{i,j} \quad \text{for all} \quad (i, j) \in B. \tag{6}$$

Lemma 1. Let X be an arbitrary $n \times m$ matrix with elements from $[0,1]$ ($X \in M_{n\times m}[0,1]$). Then $TX \in M_{n\times m}[0,1]$, i.e. the matrix set $M_{n\times m}[0,1]$ is closed with respect to the operator T.

Proof. Equation (1) immediately implies nonnegativity of elements $(TX)_{i,j} \in \mathbb{R}$ ($(i, j) \in \Omega$). For upper bounding $(TX)_{i,j} \in \mathbb{R}$ apply relation (1) given condition (3):

$$(TX)_{i,j} = \sum_{k=1}^{n}\sum_{l=1}^{m} T_{i,j,k,l} X_{k,l} \le \sum_{k=1}^{n}\sum_{l=1}^{m} T_{i,j,k,l} = 1,$$

which proves the statement of the lemma. □

The operator T can be visualized as a labelled directed impact graph G_T with vertices corresponding to elements $(i, j) \in \Omega$: a directed edge leads from the vertex $(k,l) \in \Omega$ to the vertex $(i, j) \in \Omega$ if and only if $T_{i,j,k,l} > 0$, this edge is labelled by the number $T_{i,j,k,l}$. Note that the operator T in fact defines adjacency matrix G_T, deployed in 4-dimensional array for convenience.

Example 1. Consider the network on the coordinate area $\Omega = \{1,2,\dots,n\} \times \{1,2,\dots,m\}$ with the bound $B = \{(1, j),(n, j),(i,1),(i,m) : 1 \le i \le n, 1 \le j \le m\}$, the impact operator T simulates equal impact on each internal element by its 4 neighbours:

$$T_{i,j,i,j} = \alpha \text{ for } 2 \le i \le n-1 \text{ and } 2 \le j \le m-1;$$

$$T_{i,j,i-1,j} = T_{i,j,i+1,j} = T_{i,j,i,j-1} = T_{i,j,i,j+1} = 0.25(1-\alpha)$$

$$\text{for } 2 \le i \le n-1 \text{ and } 2 \le j \le m-1;$$

$$T_{i,j,k,l} = 0 \text{ for } |i-k| + |j-l| \ge 2; \quad T_{i,j,k,l} = 0 \text{ for } i \in \{1,n\} \text{ or } j \in \{1,m\}.$$

Here a constant $\alpha \in (0,1)$ defines the impact value on the element by its 4 neighbours and by itself. The corresponding impact graph G_T for a case of $n = 5$, $m = 6$, $\alpha = 0.6$ is depicted in Fig. 1, vertices of boundary elements are denoted by ($\circ$).

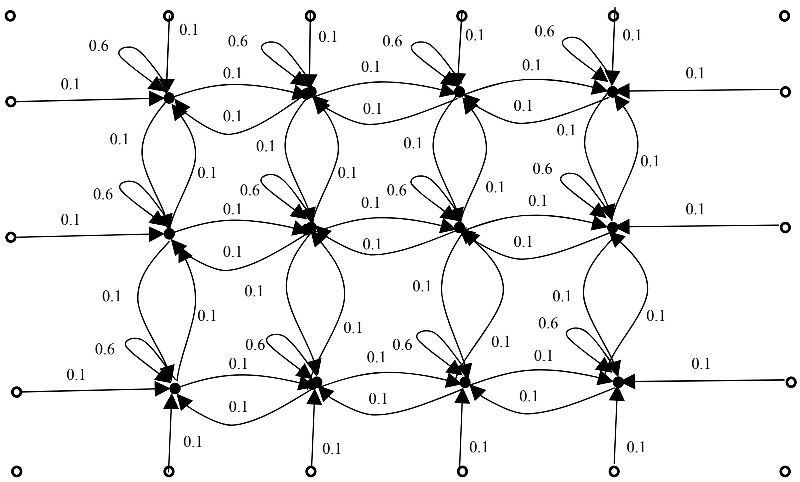

Fig. 1

Remark 1. Relations (2) and (3) in the case of $B = \varnothing$ define linear operators with stochastic (Markov) matrices (see, e.g., [7; 8]), and some its properties can be extended on a more general matrix types (see, e.g., [9]).

SPECTRUM OF THE OPERATOR T

It is well known that, although $T : M_{n\times m}[\mathbb{R}] \to M_{n\times m}[\mathbb{R}]$ is the linear operator on the linear space $M_{n\times m}[\mathbb{R}]$ (i.e. on the field of real numbers), eigenvalues and eigenvectors of the operator T in a general case are complex. Hereinafter, in the context of the operator T, usual notion 'eigenvector' is used (despite that the argument of the operator T is matrix $X \in M_{n\times m}[\mathbb{R}]$).

Lemma 2. Let $\lambda \in \mathbb{C}$ be an eigenvalue of the operator T, i.e. $TE = \lambda E$ for some nonzero $E \in M_{n\times m}[\mathbb{C}]$. Then $|\lambda| \le 1$.

Proof. Let $\left|E_{i_0,j_0}\right| = \max\limits_{(i,j)\in\Omega}\left|E_{i,j}\right|$, i.e. $\left|E_{i,j}\right|$ reaches its maximal value on $(i_0, j_0) \in \Omega$ (obviously, this maximum can be reached at several points). Then, similarly to the proof of Lemma 1, one can obtain:

$$\left|(TE)_{i_0,j_0}\right| = \left|\sum_{k=1}^{n}\sum_{l=1}^{m} T_{i_0,j_0,k,l}E_{k,l}\right| \le \sum_{k=1}^{n}\sum_{l=1}^{m} T_{i_0,j_0,k,l}\left|E_{k,l}\right| \le \left|E_{i_0,j_0}\right| \sum_{k=1}^{n}\sum_{l=1}^{m} T_{i_0,j_0,k,l} = \left|E_{i_0,j_0}\right|.$$

However, on the other hand, $(TE)_{i_0,j_0} = \lambda E_{i_0,j_0}$, thus $\left|\lambda\right| \cdot \left|E_{i_0,j_0}\right| \le \left|E_{i_0,j_0}\right|$. Therefore, since $\left|E_{i_0,j_0}\right| = \max\limits_{(i,j)\in\Omega}\left|E_{i,j}\right| \ne 0$, it yields the desired estimate $\left|\lambda\right| \le 1$. □

Theorem 1. Let $\lambda \in \mathbb{C}$ be an eigenvalue of the operator T that belongs to the unit circle ($|\lambda| = 1$), and let the impact graph G_T satisfy the following conditions:

- for any internal elements $(i_1, j_1), (i_2, j_2) \in \Omega \setminus B$ there exists a directed path from the vertex (i_1, j_1) to (i_2, j_2);
- for any internal element $(i, j) \in \Omega \setminus B$ there exists a 'loop' (an edge leading from the vertex (i, j) to the same vertex (i, j)).

Then $\lambda = 1$, and the corresponding eigenspace is generated by the eigenvector $1^{\Omega\setminus B} \in M_{n\times m}[\mathbb{C}]$ such that

$$\left(1^{\Omega\setminus B}\right)_{i,j} = \begin{cases} 1, (i,j) \in \Omega \setminus B, \\ 0, (i,j) \in B. \end{cases}$$

Proof. Let $E \in M_{n\times m}[\mathbb{C}]$ be some eigenvector that corresponds to the eigenvalue $\lambda \in \mathbb{C}$ on the unit circle ($|\lambda| = 1$). Note that $E_{i,j} = 0$ for all $(i,j) \in B$ due to relation $TE = \lambda E$.

Firstly prove that $|E_{i,j}|$ is a constant that does not depend on $(i,j) \in \Omega \setminus B$. Let $c = |E_{i_0,j_0}| = \max_{(i,j)\in\Omega} |E_{i,j}|$, i.e. $|E_{i,j}|$ reaches its maximal value on $(i_0, j_0) \in \Omega$. Since any eigenvector is nonzero by definition, constant $c = |E_{i_0,j_0}| = \max_{(i,j)\in\Omega} |E_{i,j}|$ is positive, thus $(i_0, j_0) \in \Omega \setminus B$. Using relation (1), one can obtain:

$$\lambda E_{i_0,j_0} = (TE)_{i_0,j_0} = \sum_{k=1}^{n}\sum_{l=1}^{m} T_{i_0,j_0,k,l} E_{k,l}\,. \tag{7}$$

Given normalizing conditions (2) and (3), equality (7) implies

$$|\lambda| \cdot |E_{i_0,j_0}| = |E_{i_0,j_0}| \le \sum_{k=1}^{n}\sum_{l=1}^{m} T_{i_0,j_0,k,l} |E_{k,l}| \le |E_{i_0,j_0}| \sum_{k=1}^{n}\sum_{l=1}^{m} T_{i_0,j_0,k,l} = |E_{i_0,j_0}|\,.$$

Therefore, the triangle inequality

$|E_{i_0,j_0}| = \left|\sum_{k=1}^{n}\sum_{l=1}^{m} T_{i_0,j_0,k,l} E_{k,l}\right| \le \sum_{k=1}^{n}\sum_{l=1}^{m} T_{i_0,j_0,k,l} |E_{k,l}|$ turns into equality:

$$|E_{i_0,j_0}| = \left|\sum_{k=1}^{n}\sum_{l=1}^{m} T_{i_0,j_0,k,l} E_{k,l}\right| = \sum_{k=1}^{n}\sum_{l=1}^{m} T_{i_0,j_0,k,l} |E_{k,l}|, \tag{8}$$

which is possible if and only if $|E_{k,l}| = |E_{i_0,j_0}|$ for all $(k,l) \in \Omega \setminus B$ such that $T_{i_0,j_0,k,l} > 0$. Further, recall that for nonzero $z_1, z_2 \in \mathbb{C}$ the equality $|z_1 + z_2| = |z_1| + |z_2|$ holds if and only if their arguments are equal: $\arg z_1 = \arg z_2$. So, relation (7) implies that $E_{k,l} = \lambda E_{i_0,j_0}$ for all $(k,l) \in \mathcal{E}^1(i_0, j_0)$, where the set $\mathcal{E}^1(i_0, j_0) = \{(k,l) : T_{i_0,j_0,k,l} > 0\}$ contains all elements $(k,l) \in \Omega \setminus B$ that directly impact the element (i_0, j_0) (there exists an edge from (k,l) to (i_0, j_0)). Therefore, since $(i_0, j_0) \in \mathcal{E}^1(i_0, j_0)$ (by the theorem conditions, there is the loop for the element $(i_0, j_0) \in \Omega \setminus B$), it is easy to see that $\lambda = 1$, and $E_{k,l} = E_{i_0,j_0}$ for all $(k,l) \in \mathcal{E}^1(i_0, j_0)$. Repeating these considerations for each $(\tilde{k}, \tilde{l}) \in \mathcal{E}^1(k,l)$, $(k,l) \in \mathcal{E}^1(i_0, j_0)$, one can obtain that $(\tilde{k}, \tilde{l}) \in \mathcal{E}(i_0, j_0)$, where the set $\mathcal{E}(i_0, j_0)$ contains all elements $(k,l) \in \Omega \setminus B$ that directly or indirectly impact the element (i_0, j_0) (there exists a path from (k,l) to (i_0, j_0)). Finally, the theorem conditions provide that for any $(i_1, j_1), (i_2, j_2) \in \Omega \setminus B$ there exists a directed path from

(i_1, j_1) to (i_2, j_2), thus $\mathcal{E}(i_0, j_0) = \Omega \setminus B$, $E = E_{i_0,j_0} \cdot 1^{\Omega\setminus B}$, which completes the proof of the theorem.□

Corollary. Under the conditions of Theorem 1, the eigenvalue $\lambda = 1$ of the operator T is a simple one (a single root of its characteristic polynomial).

Proof. Assume that the eigenvalue $\lambda = 1$ is not simple, i.e. it is a root of the characteristic polynomial of multiplicity 2 or more. Then, by Theorem 1, the eigenvalue $\lambda = 1$ corresponds to a one-dimensional eigenspace, and for $\lambda = 1$ there exists (see, e.g., [10, 11]) a generalized eigenvector $\widetilde{1}^{\Omega\setminus B} \in M_{n\times m}[\mathbb{C}]$: $T \cdot \widetilde{1}^{\Omega\setminus B} = \lambda \cdot \widetilde{1}^{\Omega\setminus B} + 1^{\Omega\setminus B} = \widetilde{1}^{\Omega\setminus B} + 1^{\Omega\setminus B}$. So, for an arbitrary $t \in \mathbb{N}$ one can obtain: $T^t \cdot \widetilde{1}^{\Omega\setminus B} = \widetilde{1}^{\Omega\setminus B} + t \cdot 1^{\Omega\setminus B}$, which contradicts to Lemma 1 for sufficiently large $t \in \mathbb{N}$. This contradiction proves that the eigenvalue $\lambda = 1$ is indeed simple. □

Theorem 1 proves that, under the given conditions, the unit circle can contain at most one eigenvalue of the operator T, namely $\lambda = 1$. However, the theorem does not exclude the case when the unit circle does not contain any eigenvalue of the operator T (by Lemma 2, in this case all eigenvalues of T are located inside the open unit disk). It is easy to derive from the proof of Theorem 1 that $\lambda = 1$ is an eigenvalue of the operator T if and only if $\sum_{(k,l)\notin\Omega\setminus B} T_{i,j,k,l} = 1$ for all $(i, j) \in \Omega \setminus B$, which, given conditions (2) and (3), is equivalent to the following condition:

$$\forall (i, j) \in \Omega \setminus B \ \forall (k,l) \in B : T_{i,j,k,l} = 0 . \tag{9}$$

Obviously, condition (9) means that the network bound is isolated from the rest of the network: no element $(k,l) \in B$ can impact any element $(i, j) \in \Omega \setminus B$. If condition (9) does not hold, there is at least one element $(k,l) \in B$ that impacts some element $(i, j) \in \Omega \setminus B$.

To simplify further analysis, consider a block structure for matrices on Ω with respect to the partition $\Omega = (\Omega \setminus B) \cup B$. For an arbitrary matrix $A \in M_{n\times m}[\mathbb{R}]$ consider a block $A_{\Omega\setminus B}$ of elements from $\Omega \setminus B$. Although the rectangle structure for area $\Omega \setminus B$ may be distorted (see, e.g., Fig. 3), one can treat $M_{\Omega\setminus B}[\mathbb{R}]$ (as well as $M_{\Omega\setminus B}[\mathbb{C}]$ if necessary) as a linear space of real (complex) 'vectors', whose entries are numbered by coordinate pairs $(i, j) \in \Omega \setminus B$. So, for the network in Fig. 3, one can obtain the following block $A_{\Omega\setminus B} \in M_{\Omega\setminus B}[\mathbb{R}]$ (vertices of boundary elements are denoted by «$\circ$»):

$$A_{\Omega\setminus B} = \begin{matrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & \circ & \circ & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \end{matrix}$$

Similarly, (provided $B \neq \varnothing$) consider the linear space $M_B[\mathbb{R}]$ and the block $A_B \in M_B[\mathbb{R}]$.

Further, define $T_{\Omega\setminus B,\Omega\setminus B} : M_{\Omega\setminus B}[\mathbb{R}] \to M_{\Omega\setminus B}[\mathbb{R}]$ as a linear operator on the block space $M_{\Omega\setminus B}[\mathbb{R}]$:

$$\forall (i,j)\in\Omega\setminus B\ \forall (k,l)\in\Omega\setminus B: (T_{\Omega\setminus B,\Omega\setminus B})_{i,j,k,l}=T_{i,j,k,l};$$
$$(T_{\Omega\setminus B,\Omega\setminus B}A_{\Omega\setminus B})_{i,j}=\sum_{(k,l)\in\Omega\setminus B}(T_{\Omega\setminus B,\Omega\setminus B})_{i,j,k,l}(A_{\Omega\setminus B})_{k,l}.$$

Similarly (provided $B\neq\varnothing$) one can define a linear operator $T_{\Omega\setminus B,B}$: $M_B[\mathbb{R}]\to M_{\Omega\setminus B}[\mathbb{R}]$:
$$\forall (i,j)\in\Omega\setminus B\ \forall (k,l)\in B: (T_{\Omega\setminus B,B})_{i,j,k,l}=T_{i,j,k,l};$$
$$(T_{\Omega\setminus B,B}A_B)_{i,j}=\sum_{(k,l)\in B}(T_{\Omega\setminus B,B})_{i,j,k,l}(A_B)_{k,l}.$$

Note that one can in a similar way define linear operators $T_{B,\Omega\setminus B}$: $M_{\Omega\setminus B}[\mathbb{R}]\to M_B[\mathbb{R}]$ and $T_{B,B}:M_B[\mathbb{R}]\to M_B[\mathbb{R}]$; however, according to definition of the bound B (condition (4)), these linear operators are zero.

Now equation (5) can be rewritten as a system:
$$\begin{cases}A_{\Omega\setminus B}(t+1)=T_{\Omega\setminus B,\Omega\setminus B}A_{\Omega\setminus B}(t)+T_{\Omega\setminus B,B}A_B(t)+\Delta_{\Omega\setminus B};\\ A_B(t+1)=\Delta_B.\end{cases}$$

Note that, by definition of the network bound, $\Delta_{i,j}=0$ in equation (5) for $(i,j)\in\Omega\setminus B$, that is $\Delta_{\Omega\setminus B}=0$, so (for $t\geq 0$)
$$\begin{cases}A_{\Omega\setminus B}(t+1)=T_{\Omega\setminus B,\Omega\setminus B}A_{\Omega\setminus B}(t)+T_{\Omega\setminus B,B}A_B(t);\\ A_B(t+1)=\Delta_B.\end{cases}$$

Further, the second equation implies $A_B(t)=\Delta_B$ for all $t\geq 1$, and for $t=0$ the initial condition corresponds to the boundary one by relation (6), so the obtained system can be written for any $t\geq 1$ as
$$\begin{cases}A_{\Omega\setminus B}(t+1)=T_{\Omega\setminus B,\Omega\setminus B}A_{\Omega\setminus B}(t)+T_{\Omega\setminus B,B}\Delta_B;\\ A_B(t+1)=\Delta_B,\end{cases}\qquad(10)$$
where $A_B(0)$ is defined by the initial conditions.

Consider system (10) in two cases: when condition (9) holds (isolated bound) and when it does not hold (non-isolated bound).

A. Suppose that condition (9) holds. Along with condition (4) it means that in the impact graph G_T all boundary elements are isolated (see, e.g., Fig. 2 and 3), vertices of boundary elements are denoted by «∘».

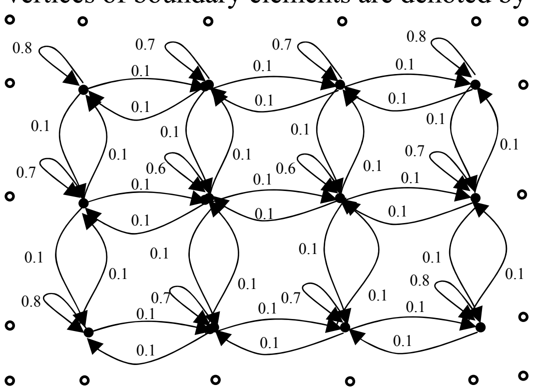

Fig. 2

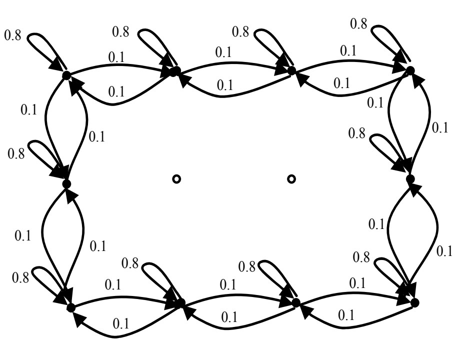

Fig. 3

In the case of isolated bound, the structure of the operator T is similar to a block-diagonal matrix: $T_{i,j,k,l} = 0$ if either $(i,j) \in \Omega \setminus B$ and $(k,l) \in B$, or $(i,j) \in B$ and $(k,l) \in \Omega \setminus B$. The operator $T_{\Omega\setminus B,B}$ is zero in the case of isolated bound, thus system (10) for $t \geq 0$ takes a form

$$\begin{cases} A_{\Omega\setminus B}(t+1) = T_{\Omega\setminus B,\Omega\setminus B} A_{\Omega\setminus B}(t); \\ A_B(t+1) = \Delta_B. \end{cases} \qquad (11)$$

Given normalizing conditions (2) and (3), the linear operator $T_{\Omega\setminus B,\Omega\setminus B}$ can be treated as an operator with stochastic matrix, which has the eigenvalue $\lambda = 1$ with the corresponding eigenvector $\mathbf{1}_{\Omega\setminus B} \in M_{\Omega\setminus B}[\mathbb{C}]$: $(\mathbf{1}_{\Omega\setminus B})_{i,j} = 1$ for all $(i,j) \in \Omega \setminus B$.

B. Suppose that condition (9) does not hold. It means that in the impact graph G_T at least one boundary element is not isolated (see Fig. 1 in Example 1).

According to condition (4), $T_{i,j,k,l} = 0$ for any $(i,j) \in B$ and $(k,l) \in \Omega$, so for the operator T it is worth considering a block structure similar to one considered in case A; in the case of non-isolated bound this structure of course is not block-diagonal.

In the case of non-isolated bound equation (5) can also be written in a general form like system (10) (but not like system (11), since the operator $T_{\Omega\setminus B,B}$ is nonzero).

Remark 2. For irreducible matrix Perron–Frobenius theorem is well known (see, e.g., [10; 12]). This theorem is similar to Theorem 1, but does not require positivity for the diagonal elements (existence of loops on the corresponding impact graph), which makes it possible for several eigenvalues to have the maximal absolute value (in the context of Theorem 1 it means that the unit circle can contain several eigenvalues of the operator T). Perron–Frobenius theorem is also applied for the Analytic Hierarchy Process, developed by T. Saaty, particularly in practice problems of economic, industrial, administrative and psychological kinds, in problems of conflict analysis and in other areas [12].

SUFFICIENT CONDITIONS FOR THE NETWORK ERGODICITY

Jordan normal form of matrix: existence of limit $\lim_{t\to+\infty} Q^t$

To analyze the network's behaviour as $t \to +\infty$, it is essentially important to know the spectral properties of the impact operator T, and these properties can be effectively explored via Jordan normal form of the corresponding matrix (see, e.g., [10; 11]). For referring convenience, consider some statements related to Jordan normal form, which are known or can be easily proven.

It is known (see, e.g., [10; 11]) that for any matrix $Q \in M_{N\times N}[\mathbb{C}]$ there exists the nondegenerate transition matrix $V \in M_{N\times N}[\mathbb{C}]$ such that $Q = VJV^{-1}$, where $J \in M_{N\times N}[\mathbb{C}]$ is the following block-diagonal matrix:

$$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_p \end{pmatrix}; \; J_s = \begin{pmatrix} \lambda_s & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_s & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_s & 1 & 0 \\ 0 & 0 & 0 & \cdots & \lambda_s & 1 \\ 0 & 0 & 0 & 0 & \cdots & \lambda_s \end{pmatrix} \in M_{N_s\times N_s}[\mathbb{C}],$$

$$\lambda_s \in \mathbb{C},\ 1 \le s \le p.$$

Matrix J is called Jordan, each matrix J_s $(1 \le s \le p)$ is called a Jordan block of dimension N_s corresponding to an eigenvalue $\lambda_s \in \mathbb{C}$; by this construction, $N_1 + N_2 + \cdots + N_p = N$. Note (see, e.g., [10; 11]) that the columns of the transition matrix V are eigenvectors and generalized eigenvectors of matrix Q, and they form so called Jordan basis in $\mathbb{C}^N$.

To compute Jordan matrix, it is convenient to use the following well–known formula:

$$J^t = \begin{pmatrix} (J_1)^t & 0 & \cdots & 0 \\ 0 & (J_2)^t & \cdots & 0 \\ \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & (J_p)^t \end{pmatrix};\ (J_s)^t = \begin{pmatrix} C_t^0 \lambda_s^t & C_t^1 \lambda_s^{t-1} & \cdots & C_t^t \lambda_s^0 & 0 \\ 0 & C_t^0 \lambda_s^t & C_t^1 \lambda_s^{t-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & C_t^0 \lambda_s^t & C_t^1 \lambda_s^{t-1} \\ 0 & 0 & \cdots & 0 & C_t^0 \lambda_s^t \end{pmatrix}$$

$$(1 \le s \le p,\ t \ge 1). \tag{12}$$

(see, e.g., [10] for approaches to defining polynomial and even analytic functions of a matrix).

Given the transition equation $Q^t = VJ^tV^{-1}$, formula (12) provides convenient tools to analyze Q^t for different $t \in \mathbb{N}$, particularly as $t \to +\infty$.

Hereinafter in the space $\mathbb{C}^N$ the norm $\|v\|_\infty = \max\limits_{1 \le j \le N} |v_j|$ ($v \in \mathbb{C}^N$) is used, in the space $M_{M \times N}[\mathbb{C}]$ the corresponding matrix norm is used:

$$\|R\|_\infty = \sup_{v \in \mathbb{C}^n : \|v\| = 1} \|Rv\|_\infty = \max_{1 \le i \le N} \sum_{k=1}^{N} |R_{i,k}|, \tag{13}$$

the norm of a linear operator is assumed to be defined by the operator norm of the corresponding matrix by formula (13).

The convergence $\lim\limits_{t \to +\infty} R_t = R$ of the matrix sequence $R_t \in M_{M \times N}[\mathbb{C}]$ ($t \in \mathbb{N}$) to matrix $R \in M_{M \times N}[\mathbb{C}]$ with respect to norm (13) is equivalent to the entrywise convergence: $\lim\limits_{t \to +\infty} (R_t)_{i,j} = R_{i,j}$ for all $1 \le i \le M$, $1 \le j \le N$; the convergence of the sequence of the linear operators is treated as the convergence of the sequence of corresponding matrices.

Lemma 3. Let J_s be a Jordan block corresponding to an eigenvalue $\lambda_s \in \mathbb{C}$ such that $|\lambda_s| < 1$. Then:

- $\lim\limits_{t \to +\infty} (J_s)^t = 0_{N_s \times N_s}$, where $0_{N_s \times N_s}$ is a zero matrix of dimension $N_s \times N_s$;
- The convergence $\lim\limits_{t \to +\infty} (J_s)^t = 0_{N_s \times N_s}$ is linear, i.e. there exist constants $C_s > 0$, $q_s \in [0,1)$ and a number $t_s \in \mathbb{N}$ such that

$$\|J_s\|_\infty \le C_s \cdot (q_s)^t \text{ for all } t \ge t_s . \tag{14}$$

Proof. It is sufficient to prove that each entry of the matrix J_s converges to zero: $0 \le C_t^i |\lambda_s|^{t-i} \le \frac{t^i}{i!} |\lambda_s|^{t-i} \xrightarrow[t\to+\infty]{} 0$, which implies the required convergence $\lim\limits_{t\to+\infty} C_t^i |\lambda_s|^{t-i} = 0$ for all $0 \le i \le N_s$ (assuming $C_t^i = 0$ for $i > t$). To prove estimate (14), one can choose sufficiently large t_s so that for $H(t) = C_t^i |\lambda_s|^{t-i}$ the following estimate holds: $q_s = \frac{H(t_s+1)}{H(t_s)} \le \left(\frac{t_s+1}{t_s}\right)^{N_s} \cdot |\lambda_s| < 1$. □

Corollary. Let the matrix Q have the simple eigenvalue $\lambda_{s_0} = 1$ with an eigenvector $v_{s_0} \in \mathbb{C}^N$, and let any other eigenvalue λ_s of Q belong to the open unit disk ($|\lambda_s| < 1$ for $s \ne s_0$). Then:

- There is the convergence $\lim\limits_{t\to+\infty} Q^t = \hat{Q} \in M_{N\times N}[\mathbb{C}]$ with $\hat{Q}v = cv_{s_0}$ for any vector $v \in \mathbb{C}^N$, where the constant $c \in \mathbb{C}$ is defined by the vector $v \in \mathbb{C}^N$;
- The convergence $\lim\limits_{t\to+\infty} Q^t = \hat{Q}$ is linear, i.e. there exist constants $C_0 > 0$, $q_0 \in [0,1)$ and a number $t_0 \in \mathbb{N}$ such that

$$\left\|Q^t - \hat{Q}\right\|_\infty \le C_0 \cdot (q_0)^t \text{ for all } t \ge t_0 . \tag{15}$$

Proof. The convergence $\lim\limits_{t\to+\infty} Q^t = \hat{Q} \in M_{N\times N}[\mathbb{C}]$ is implied by formula (12) and Lemma 3; moreover, Lemma 3 and the condition of the corollary yield equality $\hat{Q} = V\hat{J}V^{-1}$, where

$$\hat{J} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

(the only nonzero element of the matrix $\hat{J}$ corresponds to the block $J_{s_0} = J_{s_0}^t = (1)$ for all $t \ge 1$). So, $\operatorname{rank}\hat{J} = 1$, whence, due to nondegeneracy of the transition matrix V, $\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{J} = 1$. Therefore, the matrix $\hat{Q}$ defines the linear mapping with one-dimensional image generated by the vector v_{s_0}, so equality $\hat{Q}v = cv_{s_0}$ holds for some constant $c \in \mathbb{C}$. Finally,

$$\left\|Q^t - \hat{Q}\right\|_\infty = \left\|V(J^t - \hat{J})V^{-1}\right\|_\infty \le \|V\|_\infty \cdot \left\|V^{-1}\right\|_\infty \cdot \left\|J^t - \hat{J}\right\|_\infty =$$

$$= \|V\|_\infty \cdot \left\|V^{-1}\right\|_\infty \cdot \max_{s\ne s_0} \left\|(J_s)^t\right\|_\infty = \|V\|_\infty \cdot \left\|V^{-1}\right\|_\infty \cdot \max_{s\ne s_0} (C_s \cdot (q_s)^t)$$

for all $t \ge t_0 = \max_{s \ne s_0} t_s$, that proves estimate (15) and thereby completes the proof of the corollary. □

Sufficient conditions for the network ergodicity: isolated bound

In the case of isolated bound, the network's behaviour is completely described by system (11).

Theorem 2. Let the impact graph G_T satisfy the following conditions:

- for any internal elements $(i_1, j_1), (i_2, j_2) \in \Omega \setminus B$ there exists a directed path from the vertex (i_1, j_1) to (i_2, j_2);
- for any internal element $(i, j) \in \Omega \setminus B$ there is a loop (an edge leading from the vertex (i, j) to the same vertex (i, j));
- all boundary elements $(i, j) \in B$ are isolated vertices.

Then:

- $A_{\Omega\setminus B}(t) \xrightarrow[t\to+\infty]{} c \cdot \mathbf{1}_{\Omega\setminus B}$, where the constant $c \in [0,1)$ is defined by the block $A_{\Omega\setminus B}(0) \in M_{\Omega\setminus B}[0,1]$ representing the states of internal elements at the initial time $t = 0$, $\mathbf{1}_{\Omega\setminus B} \in M_{\Omega\setminus B}[0,1]$, $\left(\mathbf{1}_{\Omega\setminus B}\right)_{i,j} = 1$ for all $(i, j) \in \Omega \setminus B$;
- The convergence $A_{\Omega\setminus B}(t) \xrightarrow[t\to+\infty]{} c \cdot \mathbf{1}_{\Omega\setminus B}$ is linear, i.e. there exist constants $C_0 > 0$, $q_0 \in [0,1)$ and a number $t_0 \in \mathbb{N}$ such that

$$\left\| A_{\Omega\setminus B}(t) - c \cdot \mathbf{1}_{\Omega\setminus B} \right\|_\infty \le C_0 \cdot (q_0)^t \quad \text{for all} \quad t \ge t_0 .$$

Proof. The statement of the theorem is implied by Theorem 1 and corollary of Lemma 3. □

Theorem 2 states that (under the given conditions) for the network with isolated bound there is a set of steady states $c \cdot \mathbf{1}_{\Omega\setminus B}$ ($c \in [0,1)$), where $\mathbf{1}_{\Omega\setminus B}$ under the given conditions (see also Theorem 1) is an eigenvector of the operator $T_{\Omega\setminus B,\Omega\setminus B}$ corresponding to the eigenvalue $\lambda = 1$. Recall that the linear operator $T_{\Omega\setminus B,\Omega\setminus B}$ under the given conditions can be treated as an operator with stochastic matrix, which always has the eigenvalue $\lambda = 1$ with the eigenvector $\mathbf{1}_{\Omega\setminus B} \in M_{\Omega\setminus B}[\mathbb{R}]$ (since the initial and boundary conditions are located inside the line segment $[0,1]$, one can choose $\mathbf{1}_{\Omega\setminus B} \in M_{\Omega\setminus B}[0,1]$).

Remark 3. Under the fixed initial conditions $A(0) \in M_\Omega[0,1]$ (or equivalently, $A_{\Omega\setminus B}(0) \in M_{\Omega\setminus B}[0,1]$), computation of the steady state $c \cdot \mathbf{1}_{\Omega\setminus B}$ (in fact, it means computation of the constant $c \in [0,1)$) can be reduced through decomposition of $A_{\Omega\setminus B}(0)$ by the Jordan basis of the operator $T_{\Omega\setminus B,\Omega\setminus B}$. However, computation of the Jordan basis for real-world networks can become significantly more complicated due to the large size of the set $\Omega \setminus B$ and (consequently) the dimension of the space $M_{\Omega\setminus B}[\mathbb{R}]$. Therefore, practically reasonable approach is to approximate (numerically) the eigenvector $c \cdot \mathbf{1}_{\Omega\setminus B}$ using the iterative procedure described by system (11).

Obviously, in the case of isolated bound the block of internal elements of the network can be considered as a network with empty bound ($B = \varnothing$), and this network's behaviour is described by the first equation of system (11).

Example 2. Consider the network on the coordinate area $\Omega = \{1,2,\dots,n\} \times \times \{1,2,\dots,m\}$ with the bound $B = \varnothing$, the impact operator T simulates equal impact on internal elements by its 4 neighbours:

$$T_{i,j,i,j} = \alpha \quad \text{for all } 1 \le i \le n \text{ and } 1 \le j \le m;$$

$$T_{i,j,i\pm1,j} = 0.25(1-\alpha), \text{ if } 2 \le i \le n-1 \text{ and } 1 \le j \le m;$$

$$T_{i,j,i,j\pm1} = 0.25(1-\alpha), \text{ if } 1 \le i \le n \text{ and } 2 \le j \le m-1;$$

$$T_{1,j,2,j} = T_{n,j,n-1,j} = 0.5(1-\alpha), \text{ if } 1 \le j \le m;$$

$$T_{i,1,i,2} = T_{i,m,i,m-1} = 0.5(1-\alpha), \text{ if } 1 \le i \le n;$$

$$T_{i,j,k,l} = 0, \text{ if } |i-k| + |j-l| \ge 2,$$

where $\alpha \in (0,1)$ is a fixed constant. The corresponding impact graph G_T for the case of $n = 3$, $m = 4$, $\alpha = 0.6$ is depicted in Fig. 4.

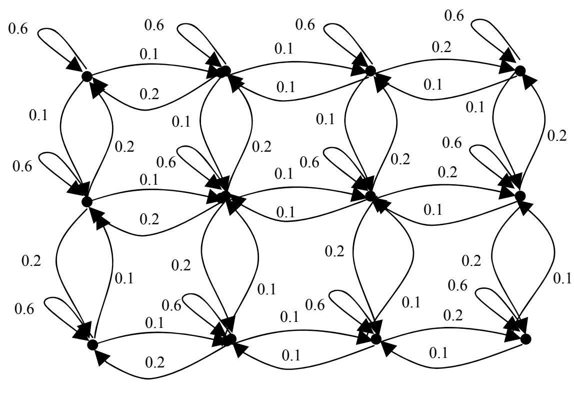

Fig. 4

Rewrite equation for transition of state $A(t)$ to the next time moment:

$$(A(t+1))_{i,j} = \alpha(A(t))_{i,j} + 0.25(1-\alpha)((A(t))_{i-1,j} +$$

$$+ (A(t))_{i+1,j} + (A(t))_{i,j-1} + (A(t))_{i,j+1}) \text{ for } 2 \le i \le n-1 \text{ and } 2 \le j \le m-1;$$

$$(A(t+1))_{1,j} = \alpha(A(t))_{1,j} + 0.25(1-\alpha)((A(t))_{1,j-1} +$$

$$+ (A(t))_{1,j+1}) + 0.5(1-\alpha)(A(t))_{2,j} \text{ for } 2 \le j \le m-1;$$

$$(A(t+1))_{n,j} = \alpha(A(t))_{n,j} + 0.25(1-\alpha)((A(t))_{n,j-1} +$$

$$+(A(t))_{n,j+1})+0.5(1-\alpha)(A(t))_{n-1,j} \text{ for } 2\le j\le m-1;$$

$$(A(t+1))_{i,1}=\alpha(A(t))_{i,1}+0.25(1-\alpha)((A(t))_{i-1,1}+(A(t))_{i+1,1})+$$

$$+0.5(1-\alpha)(A(t))_{i,2} \text{ for } 2\le i\le n-1;\ (A(t+1))_{i,n}=\alpha(A(t))_{i,n}+$$

$$+0.25(1-\alpha)((A(t))_{i-1,n}+(A(t))_{i+1,n})+0.5(1-\alpha)(A(t))_{i,n-1} \text{ for } 2\le i\le n-1;$$

$$(A(t+1))_{1,1}=\alpha(A(t))_{1,1}+0.5(1-\alpha)((A(t))_{1,2}+(A(t))_{2,1});$$

$$(A(t+1))_{n,1}=\alpha(A(t))_{n,1}+0.5(1-\alpha)((A(t))_{n,2}+(A(t))_{n-1,1});$$

$$(A(t+1))_{1,m}=\alpha(A(t))_{1,m}+0.5(1-\alpha)((A(t))_{1,m-1}+(A(t))_{2,m});$$

$$(A(t+1))_{n,m}=\alpha(A(t))_{n,m}+0.5(1-\alpha)((A(t))_{n,m-1}+(A(t))_{n-1,m}).$$

Considering coefficients of $(A(t))_{i,j}$ for the different pairs $(i,j)\in\Omega$, one can construct a function $S_w : M_\Omega[0,1]\to[0,1]$ as a 'weighted' sum of $(A(t))_{i,j}$, which is a constant value for all $t\ge 0$:

$$S_w(X)=\sum_{i=2}^{n-1}\sum_{j=2}^{m-1}X_{i,j}+0.5\left(\sum_{j=2}^{m-1}X_{1,j}+\sum_{j=2}^{m-1}X_{n,j}+\sum_{i=2}^{m-1}X_{i,1}+\sum_{i=2}^{m-1}X_{i,m}\right)+$$

$$+0.25(X_{1,1}+X_{n,1}+X_{1,m}+X_{n,m}) \text{ for } X\in M_\Omega[0,1].$$

For $X=A(t+1)$ one can obtain:

$$S_w(A(t+1))=\sum_{i=2}^{n-1}\sum_{j=2}^{m-1}(A(t+1))_{i,j}+$$

$$+0.5\left(\sum_{j=2}^{m-1}((A(t+1))_{1,j}+(A(t+1))_{n,j})+\sum_{i=2}^{n-1}((A(t+1))_{i,1}+(A(t+1))_{i,m})\right)+$$

$$+0.25((A(t+1))_{1,1}+(A(t+1))_{n,1}+(A(t+1))_{1,m}+(A(t+1))_{n,m}).$$

Simplify separately three summands in the right-hand side of the obtained relation:

$$\sum_{i=2}^{n-1}\sum_{j=2}^{m-1}(A(t+1))_{i,j}=$$

$$=\sum_{i=2}^{n-1}\sum_{j=2}^{m-1}(\alpha(A(t))_{i,j}+0.25(1-\alpha)((A(t))_{i-1,j}+(A(t))_{i+1,j}+(A(t))_{i,j-1}+(A(t))_{i,j+1}))=$$

$$=\alpha\sum_{i=2}^{n-1}\sum_{j=2}^{m-1}(A(t))_{i,j}+$$

$$+0.25(1-\alpha)\sum_{i=2}^{n-1}\sum_{j=2}^{m-1}((A(t))_{i-1,j}+(A(t))_{i+1,j}+(A(t))_{i,j-1}+(A(t))_{i,j+1})=$$

$$=\alpha\sum_{i=2}^{n-1}\sum_{j=2}^{m-1}(A(t))_{i,j}+0.25(1-\alpha)\times$$

$$\times\left(\sum_{i=1}^{n-2}\sum_{j=2}^{m-1}(A(t))_{i,j}+\sum_{i=3}^{n}\sum_{j=2}^{m-1}(A(t))_{i,j}+\sum_{i=2}^{n-1}\sum_{j=1}^{m-2}(A(t))_{i,j}+\sum_{i=2}^{n-1}\sum_{j=3}^{m}(A(t))_{i,j}\right)=$$

$$=\alpha\sum_{i=2}^{n-1}\sum_{j=2}^{m-1}(A(t))_{i,j}+0.5(1-\alpha)\left(\sum_{i=2}^{n-1}\sum_{j=2}^{m-1}(A(t))_{i,j}+\sum_{i=2}^{n-1}\sum_{j=2}^{m-1}(A(t))_{i,j}\right)+$$

$$+0.25(1-\alpha)\left(\sum_{j=2}^{m-1}(A(t))_{1,j}-\sum_{j=2}^{m-1}(A(t))_{n-1,j}-\sum_{j=2}^{m-1}(A(t))_{2,j}+\sum_{j=2}^{m-1}(A(t))_{n,j}\right)+$$

$$+0.25(1-\alpha)\left(\sum_{i=2}^{n-1}(A(t))_{i,1}-\sum_{i=2}^{n-1}(A(t))_{i,m-1}-\sum_{i=2}^{n-1}(A(t))_{i,2}+\sum_{i=2}^{n-1}(A(t))_{i,m}\right)=$$

$$=\sum_{i=2}^{n-1}\sum_{j=2}^{m-1}(A(t))_{i,j}+0.25(1-\alpha)\left(\sum_{j=2}^{m-1}((A(t))_{1,j}-(A(t))_{n-1,j})+\sum_{j=2}^{m-1}((A(t))_{n,j}-(A(t))_{2,j})\right)+$$

$$+0.25(1-\alpha)\left(\sum_{i=2}^{n-1}((A(t))_{i,1}-(A(t))_{i,m-1})+\sum_{i=2}^{n-1}((A(t))_{i,m}-(A(t))_{i,2})\right);$$

$$0.5\left(\sum_{j=2}^{m-1}(A(t+1))_{1,j}+\sum_{j=2}^{m-1}(A(t+1))_{n,j}+\sum_{i=2}^{n-1}(A(t+1))_{i,1}+\sum_{i=2}^{n-1}(A(t+1))_{i,m}\right)=$$

$$=0.5\alpha\left(\sum_{j=2}^{m-1}((A(t))_{1,j}+(A(t))_{n,j})+\sum_{i=2}^{n-1}((A(t))_{i,1}+(A(t))_{i,m})\right)+$$

$$+0.125(1-\alpha)\sum_{j=2}^{m-1}((A(t))_{1,j-1}+(A(t))_{1,j+1}+(A(t))_{n,j-1}+(A(t))_{n,j+1})+$$

$$+0.125(1-\alpha)\sum_{i=2}^{n-1}((A(t))_{i-1,1}+(A(t))_{i+1,1}+((A(t))_{i-1,m}+(A(t))_{i+1,m}))+$$

$$+0.25(1-\alpha)\left(\sum_{j=2}^{m-1}((A(t))_{2,j}+(A(t))_{n-1,j})+\sum_{i=2}^{n-1}((A(t))_{i,2}+(A(t))_{i,m-1})\right)=$$

$$=0.5\alpha\left(\sum_{j=2}^{m-1}((A(t))_{1,j}+(A(t))_{n,j})+\sum_{i=2}^{n-1}((A(t))_{i,1}+(A(t))_{i,m})\right)+$$

$$+0.25(1-\alpha)\left(\sum_{j=2}^{m-1}((A(t))_{1,j}+(A(t))_{n,j})+\sum_{i=2}^{n-1}((A(t))_{i,1}+(A(t))_{i,m})\right)+$$

$$+0.125(1-\alpha)((A(t))_{1,1}-(A(t))_{1,m-1}-(A(t))_{1,2}+(A(t))_{1,m})+$$

$$+0.125(1-\alpha)((A(t))_{n,1}-(A(t))_{n,m-1}-(A(t))_{n,2}+(A(t))_{n,m})+$$

$$+0.125(1-\alpha)((A(t))_{1,1}-(A(t))_{n-1,1}-(A(t))_{2,1}+(A(t))_{n,1})+$$

$$+0.125(1-\alpha)((A(t))_{1,m}-(A(t))_{n-1,m}-(A(t))_{2,m}+(A(t))_{n,m})+$$

$$+0.25(1-\alpha)\left(\sum_{j=2}^{m-1}((A(t))_{2,j}+(A(t))_{n-1,j})+\sum_{i=2}^{n-1}((A(t))_{i,2}+(A(t))_{i,m-1})\right)=$$

$$=0.25\left(\sum_{j=2}^{m-1}((A(t))_{1,j}+(A(t))_{n,j})+\sum_{i=2}^{n-1}((A(t))_{i,1}+(A(t))_{i,m})\right)+$$

$$+0.25\alpha\left(\sum_{j=2}^{m-1}((A(t))_{1,j}+(A(t))_{n,j})+\sum_{i=2}^{n-1}((A(t))_{i,1}+(A(t))_{i,m})\right)+$$

$$+0.25(1-\alpha)\left(\sum_{j=2}^{m-1}((A(t))_{2,j}+(A(t))_{n-1,j})+\sum_{i=2}^{n-1}((A(t))_{i,2}+(A(t))_{i,m-1})\right)+$$

$$+0.125(1-\alpha)((A(t))_{1,1}-(A(t))_{1,m-1}-(A(t))_{1,2}+(A(t))_{1,m})+$$

$$+0.125(1-\alpha)((A(t))_{n,1}-(A(t))_{n,m-1}-(A(t))_{n,2}+(A(t))_{n,m})+$$

$$+0.125(1-\alpha)((A(t))_{1,1}-(A(t))_{n-1,1}-(A(t))_{2,1}+(A(t))_{n,1})+$$

$$+0.125(1-\alpha)((A(t))_{1,m}-(A(t))_{n-1,m}-(A(t))_{2,m}+(A(t))_{n,m});$$

$$0.25((A(t+1))_{1,1}+(A(t+1))_{n,1}+(A(t+1))_{1,m}+(A(t+1))_{n,m})=$$

$$=0.25\alpha((A(t))_{1,1}+(A(t))_{n,1}+(A(t))_{1,m}+(A(t))_{n,m})+$$

$$+0.125(1-\alpha)((A(t))_{1,2}+(A(t))_{2,1}+(A(t))_{n-1,1}+(A(t))_{n,2})+$$

$$+0.125(1-\alpha)((A(t))_{2,m}+(A(t))_{1,m-1}+(A(t))_{n-1,m}+(A(t))_{n,m-1}).$$

Collecting these summands together, one can obtain:

$$S_w(A(t+1))=$$

$$=\sum_{i=2}^{n-1}\sum_{j=2}^{m-1}(A(t))_{i,j}+0.5\left(\sum_{j=2}^{m-1}((A(t))_{1,j}+(A(t))_{n,j})+\sum_{i=2}^{n-1}((A(t))_{i,1}+(A(t))_{i,m})\right)+$$

$$+0.25((A(t))_{1,1}+(A(t))_{n,1}+(A(t))_{1,m}+(A(t))_{n,m})=S_w(A(t)),$$

whence

$$S_w(A(0))=S_w\left(\lim_{t\to+\infty}A(t)\right)=S_w\left(c\cdot\mathbf{1}_{\Omega\setminus B}\right)=c\cdot S_w\left(\mathbf{1}_{\Omega\setminus B}\right)=$$

$$=c((n-2)(m-2)+0.5\cdot 2\cdot(n-2+m-2)+0.25\cdot 4)=c\cdot(n-1)(m-1),$$

i.e. $c=\frac{S_w(A(0))}{(n-1)(m-1)}$. Particularly, for $n=20$, $m=10$, $\alpha=0.8$, $(A(0))_{i,j}=$ $=\begin{cases}1, & i=j=1,\\ 0, & \text{otherwise},\end{cases}$ one can obtain $c=\frac{0.25}{(20-1)(10-1)}\approx 0.00146$. All data are written with precision up to 0.00001 which corresponds to relative error $\frac{0.00001}{0.00146}\cong 0.01$, the convergence by the iterative procedure (10) with such precision is achieved approximately for $t \ge 5000$. It is interesting to note that for this impact operator T the steady state $c \cdot 1_{\Omega \setminus B}$ does not depend on $\alpha \in (0,1)$; however, the value $\alpha \in (0,1)$ affects on the convergence rate (for $\alpha = 0.6$ precision of 0.00001 is achieved approximately for $t \ge 2500$).

Sufficient conditions for the network ergodicity: non-isolated bound

In the case of non-isolated bound, the network's behaviour is completely described by system (10).

Theorem 3. Let the impact graph G_T satisfy the following conditions:

- for any internal elements $(i_1, j_1), (i_2, j_2) \in \Omega \setminus B$ there exists a directed path from the vertex (i_1, j_1) to (i_2, j_2);
- for any internal element $(i, j) \in \Omega \setminus B$ there is a loop (an edge leading from the vertex (i, j) to the same vertex (i, j));
- at least one boundary element $(i, j) \in B$ is not an isolated vertex.

Then there exists the unique vector $\hat{A}_{\Omega \setminus B} \in M_{\Omega \setminus B}[\mathbb{R}]$ such that:

- $A_{\Omega \setminus B}(t) \xrightarrow[t \to +\infty]{} \hat{A}_{\Omega \setminus B}$;
- the convergence $A_{\Omega \setminus B}(t) \xrightarrow[t \to +\infty]{} \hat{A}_{\Omega \setminus B}$ is linear, i.e. there exist constants $C_0 > 0$, $q_0 \in [0,1)$ and a number $t_0 \in \mathbb{N}$ such that

$$\left\| A_{\Omega \setminus B}(t) - \hat{A}_{\Omega \setminus B} \right\|_\infty \le C_0 \cdot (q_0)^t \text{ for all } t \ge t_0 .$$

Proof. System (10) yields the explicit form for $A_{\Omega \setminus B}(t)$ ($t \ge 1$):

$$A_{\Omega \setminus B}(t) = (T_{\Omega \setminus B, \Omega \setminus B})^t A_{\Omega \setminus B}(0) + \sum_{s=0}^{t-1} (T_{\Omega \setminus B, \Omega \setminus B})^s T_{\Omega \setminus B, B} \Delta_B \tag{16}$$

Due to Theorem 1, all eigenvalues of the operator $T_{\Omega \setminus B, \Omega \setminus B}$ are located inside the unit disk, so by virtue of Lemma 3 there exist constants $C > 0$, $q \in [0,1)$ and a number $t_0 \in \mathbb{N}$ such that $\left\| T_{\Omega \setminus B, \Omega \setminus B} \right\|_\infty \le C \cdot q^t$ for all $t \ge t_0$, which yields the required convergence:

$$A_{\Omega \setminus B}(t) \xrightarrow[t \to +\infty]{} \hat{A}_{\Omega \setminus B} = \sum_{t=0}^{+\infty} \left(T_{\Omega \setminus B, \Omega \setminus B} \right)^t T_{\Omega \setminus B, B} \Delta_B .$$

Finally,

$$\left\| A_{\Omega \setminus B}(t) - \hat{A}_{\Omega \setminus B} \right\|_\infty = \left\| \sum_{s=t+1}^{+\infty} (T_{\Omega \setminus B, \Omega \setminus B})^s T_{\Omega \setminus B, B} \Delta_B \right\|_\infty \le C \left\| T_{\Omega \setminus B, B} \Delta_B \right\|_\infty \frac{q^{t+1}}{1-q},$$

so the convergence $A_{\Omega \setminus B}(t) \xrightarrow[t \to +\infty]{} \hat{A}_{\Omega \setminus B}$ is indeed linear. □

Theorem 3 proves that, under the given conditions, for the case of non-isolated bound there exists the unique steady state $\hat{A}_{\Omega \setminus B} \in M_{\Omega \setminus B}[\mathbb{R}]$ (moreover: $\hat{A}_{\Omega \setminus B} \in M_{\Omega \setminus B}[0,1]$, since the initial and boundary conditions are located inside the line segment $[0,1]$). The equation for the state $\hat{A}_{\Omega\setminus B}$ can be obtained from the first equation of system (10) as $t \to +\infty$:

$$\hat{A}_{\Omega\setminus B} = T_{\Omega\setminus B,\Omega\setminus B}\hat{A}_{\Omega\setminus B} + T_{\Omega\setminus B,B}\Delta_B . \qquad (17)$$

Remark 4. Since all eigenvalues of the operator $T_{\Omega\setminus B,\Omega\setminus B}$ under the given conditions are located inside the open unit disk, equation (17) under these conditions has the unique solution. However, direct solving equation (17) for real-world networks usually becomes significantly more complicated due to the large size of the set $\Omega \setminus B$ and (consequently) the large dimension of the space $M_{\Omega\setminus B}[\mathbb{R}]$. Therefore, practically reasonable approach is to approximate (numerically) $\hat{A}_{\Omega\setminus B}$ using the iterative procedure described by system (10).

Example 3. The network with the impact operator T from Example 1 is obviously the network with non-isolated bound. The conditions of Theorem 3 hold, so the network has the unique steady state $\hat{A}_{\Omega\setminus B} \in M_{\Omega\setminus B}[0,1]$. Equation (17) for this network takes the form:

$$(\hat{A}_{\Omega\setminus B})_{i,j} = 0.25(1-\alpha)((\hat{A}_{\Omega\setminus B})_{i-1,j} +$$

$$+ (\hat{A}_{\Omega\setminus B})_{i+1,j} + (\hat{A}_{\Omega\setminus B})_{i,j-1} + (\hat{A}_{\Omega\setminus B})_{i,j+1}) + \alpha(\hat{A}_{\Omega\setminus B})_{i,j}$$

for all internal $(i,j) \in \Omega \setminus B$ (i.e., for all $2 \le i \le n-1$ and $2 \le j \le m-1$). Thus, given $\alpha \ne 1$, one can obtain:

$$(\hat{A}_{\Omega\setminus B})_{i,j} = \tfrac{1}{4}((\hat{A}_{\Omega\setminus B})_{i-1,j} + (\hat{A}_{\Omega\setminus B})_{i+1,j} + (\hat{A}_{\Omega\setminus B})_{i,j-1} + (\hat{A}_{\Omega\setminus B})_{i,j+1}) .$$

Assume that the block $A_B = \hat{A}_B = \Delta_B$ representing the states of boundary elements of the network is defined by four arithmetic progressions:

$$(\Delta_B)_{1,j} = (\Delta_B)_{1,1} + (j-1)\tfrac{(\Delta_B)_{1,m} - (\Delta_B)_{1,1}}{m-1}, \ 1 \le j \le m;$$

$$(\Delta_B)_{n,j} = (\Delta_B)_{n,1} + (j-1)\tfrac{(\Delta_B)_{n,m} - (\Delta_B)_{n,1}}{m-1}, \ 1 \le j \le m;$$

$$(\Delta_B)_{i,1} = (\Delta_B)_{1,1} + (i-1)\tfrac{(\Delta_B)_{n,1} - (\Delta_B)_{1,1}}{n-1}, \ 1 \le i \le n;$$

$$(\Delta_B)_{i,m} = (\Delta_B)_{1,m} + (i-1)\tfrac{(\Delta_B)_{n,m} - (\Delta_B)_{1,m}}{n-1}, \ 1 \le i \le n,$$

where the states of the corner elements $b_{\text{top, left}} = (\Delta_B)_{1,1}$, $b_{\text{top, right}} = (\Delta_B)_{1,m}$, $b_{\text{bottom, left}} = (\Delta_B)_{n,1}$, $b_{\text{bottom, right}} = (\Delta_B)_{n,1}$ are the given constants from $[0,1]$. It is easy to see that all elements of matrix $\hat{A}_\Omega \in M_\Omega[0,1]$ (the steady state of the network) also form arithmetic progressions by each row and each column:

$$(\hat{A}_\Omega)_{i,j} = b_{\text{top, left}} + (i-1)\frac{b_{\text{bottom, left}} - b_{\text{top, left}}}{n-1} +$$

$$+ (j-1)\frac{(b_{\text{top, right}} - b_{\text{top, left}}) + (i-1)\frac{(b_{\text{bottom, right}} - b_{\text{bottom, left}}) - (b_{\text{top, right}} - b_{\text{top, left}})}{n-1}}{m-1} =$$

$$= b_{\text{top, left}} + \tfrac{i-1}{n-1}(b_{\text{bottom, left}} - b_{\text{top, left}}) + \tfrac{j-1}{m-1}(b_{\text{top, right}} - b_{\text{top, left}}) +$$

$$+\frac{(i-1)(j-1)}{(n-1)(m-1)}((b_{\text{bottom, right}}-b_{\text{bottom, left}})-(b_{\text{top, right}}-b_{\text{top, left}})) \quad (18)$$

The table contains the steady state $\hat{A}_{\Omega}$ of the given network for $n=20$, $m=10$, $\alpha=0.8$, $b_{\text{top, left}}=0.3$, $b_{\text{top, right}}=0.5$, $b_{\text{bottom, left}}=0.8$, $b_{\text{bottom, right}}= =0.9$; all data are written with precision up to 0.01. Precision 0.01 is achieved by the iterative procedure (10) approximately for $t\geq 650$. It is interesting to note that for this impact operator T the steady state $\hat{A}_{\Omega}$ does not depend on $\alpha\in(0,1)$; however, the value $\alpha\in(0,1)$ affects on the convergence rate (for $\alpha=0.6$ precision of 0.01 is achieved approximately for $t\geq 320$).

T a b l e

i	j									
	1	2	3	4	5	6	7	8	9	10
1	0.30	0.32	0.34	0.37	0.39	0.41	0.43	0.46	0.48	0.50
2	0.33	0.35	0.37	0.39	0.41	0.43	0.46	0.48	0.50	0.52
3	0.35	0.37	0.39	0.42	0.44	0.46	0.48	0.50	0.52	0.54
4	0.38	0.40	0.42	0.44	0.46	0.48	0.50	0.52	0.54	0.56
5	0.41	0.43	0.45	0.46	0.48	0.50	0.52	0.54	0.56	0.58
6	0.43	0.45	0.47	0.49	0.51	0.53	0.55	0.57	0.59	0.61
7	0.46	0.48	0.50	0.51	0.53	0.55	0.57	0.59	0.61	0.63
8	0.48	0.50	0.52	0.54	0.56	0.57	0.59	0.61	0.63	0.65
9	0.51	0.53	0.55	0.56	0.58	0.60	0.62	0.63	0.65	0.67
10	0.54	0.55	0.57	0.59	0.60	0.62	0.64	0.66	0.67	0.69
11	0.56	0.58	0.60	0.61	0.63	0.65	0.66	0.68	0.69	0.71
12	0.59	0.61	0.62	0.64	0.65	0.67	0.68	0.70	0.72	0.73
13	0.62	0.63	0.65	0.66	0.68	0.69	0.71	0.72	0.74	0.75
14	0.64	0.66	0.67	0.69	0.70	0.72	0.73	0.74	0.76	0.77
15	0.67	0.68	0.70	0.71	0.72	0.74	0.75	0.77	0.78	0.79
16	0.69	0.71	0.72	0.74	0.75	0.76	0.78	0.79	0.80	0.82
17	0.72	0.73	0.75	0.76	0.77	0.79	0.80	0.81	0.82	0.84
18	0.75	0.76	0.77	0.78	0.80	0.81	0.82	0.83	0.85	0.86
19	0.77	0.79	0.80	0.81	0.82	0.83	0.84	0.86	0.87	0.88
20	0.80	0.81	0.82	0.83	0.84	0.86	0.87	0.88	0.89	0.90

Remark 5. In Examples 2 and 3 it is possible to compute analytically the steady state as $t\to+\infty$. However, as it is mentioned in Remark 4, direct solving equation (17) for real-world networks usually becomes significantly more complicated due to the large dimension of the space $M_{\Omega\backslash B}[\mathbb{R}]$. Therefore, practically reasonable is to apply the iterative procedure described by system (10). For more details about analytical solving recurrent relations with multiple indices (with indices $(i,j)\in\Omega$ in the given case of two-dimensional network) see, e.g., [13].

CONCLUSIONS

- Matrix model for social network is proposed, mutual impact of network elements is represented by the linear impact operator T and the corresponding labelled directed impact graph G_T.

- Sufficient conditions for the network ergodicity are given in terms of existence of a steady state, which defines the network's behaviour as $t \to +\infty$.
- For the proposed model, spectral properties of the operator T are explored.
- Sufficient conditions for the network ergodicity are given in the form of existing eigenvalues of the operator T on the unit circle, and in the form of strong connectivity of the impact graph G_T.

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МАТРИЧНО-ГРАФІЧНЕ МОДЕЛЮВАННЯ СОЦІАЛЬНОЇ МЕРЕЖІ: ЕРГОДИЧНІ ВЛАСТИВОСТІ / І.Я. Спекторський, В.М. Статкевич, О.В. Стусь

Анотація. Запропоновано математичний апарат моделювання соціальних мереж, який дозволяє отримати достатні умови ергодичності мережі, тобто існування граничного стаціонарного стану при $t \to +\infty$. Запропонована модель є лінійною: елементи мережі утворюють двовимірний масив (матрицю), елементом матриці в момент часу $t \ge 0$ є стан $A_{i,j}(t) \in [0,1]$ елемента з координатами $(i,j) \in \Omega = \{1,2,\dots,n\} \times \{1,2,\dots,m\}$. Взаємний вплив між елементами задано оператором впливу T — чотиривимірним масивом, елементи $T_{i,j,k,l} \ge 0$ якого позначають вплив елемента $(k,l) \in \Omega$ на елемент $(i,j) \in \Omega$: $(TA)_{i,j} = \sum_{k=1}^{n} \sum_{l=1}^{m} T_{i,j,k,l} A_{k,l}$. Для T запропоновано зображення у вигляді графу G_T , вершини якого відповідають елементам $(i,j) \in \Omega$: орієнтоване ребро (дуга) з міткою $T_{i,j,k,l}$ веде від вершини $(k,l) \in \Omega$ до вершини $(i,j) \in \Omega$ тоді й тільки тоді, коли $T_{i,j,k,l} > 0$. На Ω виділено край $B \subset \Omega$: $T_{i,j,k,l} = 0$ для $(k,l) \in \Omega$, $(i,j) \in B$. Стан $A(t+1)$ мережі у момент часу $t+1$ визначається станом $A(t)$ мережі у момент часу $t \ge 0$ згідно з рівнянням $A(t+1) = TA(t) + \Delta$, де матриця Δ розмірності $n \times m$ визначає стан крайових елементів мережі; $\Delta_{i,j} = 0$ для внутрішніх елементів $(i,j) \in \Omega \setminus B$. Достатні умови ергодичності мережі надано у термінах властивостей зв'язності графу впливу G_T : мають існувати шляхи між довільними вершинами та усі петлі. Наведені умови забезпечують розташування спектра оператора T всередині одиничного круга за винятком, можливо, $\lambda = 1$; доведено, що $\lambda = 1$ є власним числом T лише у випадку ізольованого краю (жоден крайовий елемент не впливає на жоден внутрішній елемент мережі). Наведені спектральні властивості T забезпечують існування стаціонарного стану, який можна знаходити ітераційною процедурою $A(t+1) = TA(t) + \Delta$ за заданим $A(0)$ з геометричною (лінійною) швидкістю збіжності.

Ключові слова: соціальна мережа, моделювання, ергодичність, власне число, жорданова нормальна форма.